Asymptotic properties of realized power variations and related functionals of semimartingales

Jean Jacod

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Abstract. This paper is concerned with the asymptotic behavior of sums of the form $U^n(f)_t = \sum_{i=1}^{[t/\Delta_n]} f(X_{i\Delta_n} - X_{(i-1)\Delta_n})$, where X is a 1-dimensional semimartingale and f a suitable test function, typically $f(x) = |x|^r$, as $\Delta_n \to 0$. We prove a variety of "laws of large numbers", that is convergence in probability of $U^n(f)_t$, sometimes after normalization. We also exhibit in many cases the rate of convergence, as well as associated central limit theorems.

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1 Introduction

In many practical situations one observes a process X at finitely many times, and from these observations one wants to infer various properties of the process. For example, in finance the price of an asset is observed at discrete times and one aims to determine the volatility or the integrated volatility, or perhaps the presence of jumps and some properties about their sizes. In statistics one wants to determine the parameters on which the law of the process depends, or one may want to perform some non-parametric inference on the model.

There are indeed two very different situations. One is when the observations occur at time $0, \Delta, 2\Delta, \cdots, n\Delta$ for a fixed time lag Δ, whereas n is large : then any kind of inference necessitates some "ergodic" properties of the basic process. Another situation is what is called *high frequency* observations, where the time lag Δ is small, which in the asymptotic setting means that we let $\Delta = \Delta_n$ depend on the number n of observations and go to 0 as $n \to \infty$. This second situation is the one we are interested in here.

A first, well known, example of how discrete observations of X allow to approximate some basic characteristics of the process is the convergence of the "realized" (or approximate) quadratic variation towards the "true" one. More generally one may look at the

*Institut de Mathématiques de Jussieu, 175 rue du Chevaleret 75 013 Paris, France (CNRS – UMR 7586, and Université Pierre et Marie Curie - P6)

realized r-th power variation at stage n, that is the (observable) process

$$\{X\}_t^{r,n} = \sum_{i=1}^{[t/\Delta_n]} |X_{i\Delta_n} - X_{(i-1)\Delta_n}|^r. \tag{1.1}$$

When $r = 2$ the processes $\{X\}^{2,n}$ converge (as $\Delta_n \to 0$) to $[X, X]$, the quadratic variation of X, as soon as X is a semimartingale, and even in some more general situations. When $r > 2$ then $\{X\}_t^{r,n}$ converges to $\sum_{s\leq t} |\Delta X_s|^r$ (where ΔX_s is the size of the jump of X at time s) for any semimartingale, also an old result due to Lépingle in [10]. When $r \in (0, 2)$ then $\{X\}_t^{r,n}$ blows up in general, but $\Delta_n^{1-r/2}\{X\}_t^{r,n}$ converges to the continuous part of $[X, X]_t$: this does not hold in general, though, but under some (weak) assumptions on X. So this allows in principle to "separate" the jumps of X from its continuous part.

Again for practical applications, having the convergence of $\{X\}^{r,n}$ (possibly after normalization) is not enough, we need rates and, if possible, an associated central limit theorem. This describes the main aim of this paper: find conditions for the above convergence, and for associated CLTs when they exist. We do that for the processes $\{X\}^{r,n}$, and more generally for the following processes, for suitable test functions f and cut-off exponent $\varpi > 0$ and level $\alpha > 0$:

$$\left.\begin{array}{l} V^n(f)_t = \sum_{i=1}^{[t/\Delta_n]} f(X_{i\Delta_n} - X_{(i-1)\Delta_n}), \\ V'^n(f)_t = \sum_{i=1}^{[t/\Delta_n]} f((X_{i\Delta_n} - X_{(i-1)\Delta_n})/\sqrt{\Delta_n}), \\ V''^n(\varpi, \alpha)_t = \sum_{i=1}^{[t/\Delta_n]} (X_{i\Delta_n} - X_{(i-1)\Delta_n})^2 1_{\{|X_{i\Delta_n} - X_{(i-1)\Delta_n}| \leq \alpha\Delta_n^{\varpi}\}}\,. \end{array}\right\} \tag{1.2}$$

The convergence of these processes and the associated CLTs hold or not, depending on the properties of f of course, and especially on its behavior near 0, but also on the properties of the basic semimartingale X. Note that we *always assume that* $\Delta_n \to 0$.

The reader may find motivations and practical uses of realized power variations in finance in Andersen, Bollersley and Diebold [2] or Barndorff-Nielsen and Shephard [3] and references therein, for continuous processes. The later authors have also introduced and thoroughly used the "bi-power variations" where the summands in (1.1) are products of powers of two successive increments of X instead of one, and probably what follows can also be done for bi- or multi-power variations as well. The case where X is discontinuous has been studied by Mancini [11], [12] (using processes similar to $V''^n(\varpi, \alpha)$) and Woerner [14], [15] (for the power and bi-power variations) and recently by Barndorff-Nielsen, Shephard and Winkel [5], and in those papers special cases of the forthcoming results may be found.

In [8] we have considered the same problems than here when X is a Lévy processes, with almost complete answers. In the semimartingale case the picture shown below is neither as good nor as complete as in the Lévy case. The proofs are mostly quite different (except for Theorem 2.2), hence this paper is essentially independent of [8] although the basic ideas are the same. On the other hand, some of the results here heavily rely upon the paper [4] in which similar problems have been solved when X is *continuous*.

Let us also mention that only the 1-dimensional case is considered here, although it covers the case where X is one of the components of a multidimensional semimartingale.

Some results obviously hold as well when X is multidimensional (those concerned with $V'^n(f)$ in particular), others do not: if f is singular at 0, the description of the singularity in the multidimensional case is clearly much more sophisticated than in dimension 1.

The main notation, assumptions and results are gathered in Section 2. All (unfortunately rather tedious) proofs are in the subsequent sections.

2 Notation, assumptions, results

2.1 Some general notation.

Let us first introduce a number of notation to be used throughout. With any process Y we associate its increments $\Delta_i^n Y$ and the "discretized process" as follows

$$\Delta_i^n Y = Y_{i\Delta_n} - Y_{(i-1)\Delta_n}, \qquad Y_t^{(n)} = Y_{\Delta_n[t/\Delta_n]} = Y_0 + \sum_{i=1}^{[t/\Delta_n]} \Delta_i^n Y. \tag{2.1}$$

As soon as Y is càdlàg (= right continuous with left limits), we have $Y^{(n)} \xrightarrow{\text{Sk}} Y$ (ω-wise convergence for the Skorokhod topology). If a process Y belongs to the set $\mathcal{V}$ of all processes of locally of finite variation, we denote by $v(Y)_t = \int_0^t |dY_s|$ its " variation process".

Next we give a series of notational conventions for the convergence of a sequence (Y^n) of (càdlàg) processes; below, α_n is a sequence of positive, possibly random, numbers:

- $Y^n \xrightarrow{\text{u.c.p.}} Y$ or $Y_t^n \xrightarrow{\text{u.c.p.}} Y_t$ (or, *converges u.c.p.*) , if $\sup_{s\leq t} |Y_s^n - Y_s| \xrightarrow{\mathbb{P}} 0$ for all $t > 0$;
- $Y^n \xrightarrow{\text{Sk.p.}} Y$ or $Y_t^n \xrightarrow{\text{Sk.p.}} Y_t$, if the convergence takes place in probability, for the Skorokhod topology;
- $Y^n \xrightarrow{\text{v.p.}} Y$ or $Y_t^n \xrightarrow{\text{v.p.}} Y_t$ (or, *converges v.p.*) if $v(Y^n - Y)_t \xrightarrow{\mathbb{P}} 0$ for all $t > 0$;
- $Y^n \xrightarrow{\mathcal{L}-(s)} Y$ or $Y_t^n \xrightarrow{\mathcal{L}-(s)} Y_t$ if there is stable convergence in law, see below;
- $Y^n = \mathrm{o}_{Pu}(\alpha_n)$ or $Y_t^n = \mathrm{o}_{Pu}(\alpha_n)$ if $Y^n/\alpha_n \xrightarrow{\text{u.c.p.}} 0$;
- $Y^n = \mathrm{O}_{Pu}(\alpha_n)$ or $Y_t^n = \mathrm{O}_{Pu}(\alpha_n)$ if the sequences $(\sup_{s\leq t} |Y_s^n/\alpha_n|)_{n\geq 1}$ are tight;
- an array (ζ_i^n) of variables is *asymptotically negligible*, (AN) for short, if $\sum_{i=1}^{[t/\Delta_n]} \zeta_i^n \xrightarrow{\text{u.c.p.}} 0$.

When each Y^n is defined on $(\Omega, \mathcal{F}, \mathbb{P})$, recall (see e.g. [6]) that $Y^n \xrightarrow{\mathcal{L}-(s)} Y$ means that Y is a càdlàg process defined on an extension of $(\Omega, \mathcal{F}, \mathbb{P})$, and that $\mathbb{E}(Zg(Y^n)) \to \mathbb{E}(Zg(Y))$ for all bounded $\mathcal{F}$–measurable variable Z and all bounded continuous function g on the space of all càdlàg functions, endowed with the Skorokhod topology.

Throughout, the following functions h_r for $r \in (0, \infty)$ and ψ_η for $\eta \in (0, \infty]$ and ϕ_s for $s \in [0, 2]$ will often occur : we first fix a C^∞ function ψ having $1_{\{|x|\leq 1\}} \leq \psi(x) \leq 1_{\{|x|\leq 2\}}$, and then set

$$\left.\begin{array}{l} h_r(x) = |x|^r, \\ \psi_\eta(x) \;=\; \begin{cases} \psi(x/\eta) & \text{if } \eta < \infty \\ 1 & \text{if } \eta = \infty, \end{cases} \\ \phi_r(x) \;=\; \begin{cases} 1 \bigwedge |x|^r & \text{if } 0 < r < \infty \\ 1_{\mathbb{R}\backslash\{0\}}(x) & \text{if } r = 0. \end{cases} \end{array}\right\} \tag{2.2}$$

Next, we introduce several classes of functions on $\mathbb{R}$. We denote by $\mathcal{E}$ the set of all Borel functions with at most polynomial growth, and for $r \in (0, \infty)$ we denote by $\mathcal{E}_r$ and $\mathcal{E}'_r$ and $\mathcal{E}''_r$ the following sets of functions :

$$\left.\begin{array}{lll} \mathcal{E}_r & : & \text{all } f \in \mathcal{E} \text{ with } \ f(x) = |x|^r \ \text{ on a neighborhood of } 0 \\ \mathcal{E}'_r & : & \text{all } f \in \mathcal{E} \text{ with } \ f(x) \sim |x|^r \ \text{ as } x \to 0 \\ \mathcal{E}''_r & : & \text{all } f \text{ locally bounded with } \ f(x) = \mathrm{O}(|x|^r) \ \text{ as } x \to 0 \\ \mathcal{E}'''_r & : & \text{all } f \text{ locally bounded with } \ f(x) = \mathrm{o}(|x|^r) \ \text{ as } x \to 0. \end{array}\right\} \tag{2.3}$$

We write $\mathcal{E}_r^b$, $\mathcal{E}_r'^b$, $\mathcal{E}_r''^b$ and $\mathcal{E}_r'''^b$ for the sets of bounded functions belonging to $\mathcal{E}_r$, $\mathcal{E}'_r$, $\mathcal{E}''_r$ and $\mathcal{E}'''_r$ respectively. We have $\phi_r \in \mathcal{E}_r^b \cap C^0$, where as usual C^p denotes the set of p times continuously differentiable functions, resp. continuous, for $p \geq 1$, resp. $p = 0$.

Below, K is a constant which changes from line to line and may depend on X and its characteristics, and we write K_p if we want to emphasize its dependency on some parameter p. We write U for a generic $\mathcal{N}(0,1)$ variable, and $m_r = \mathbb{E}(|U|^r)$ is its rth absolute moment. We also denote by ρ_s the normal law $\mathcal{N}(0, s^2)$, and write $\rho_s(f) = \int f(x)\rho_s(dx)$.

2.2 The assumptions.

We start with a semimartingale X on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We fix a truncation function κ (bounded with compact support, with $\kappa(x) = x$ on a neighborhood of 0): this function is a priori arbitrary and usually $\kappa(x) = x1_{\{|x|\leq 1\}}$, but in this paper we suppose that it is *continuous* : this simplifies some of the assumptions below. We call (B, C, ν) its predictable characteristics : ν is the compensator of the jump measure μ of X, and $C = \langle X^c, X^c\rangle$, where X^c is the continuous martingale part of X, and B depends on the choice of κ. With $\kappa'(x) = x - \kappa(x)$, we then have

$$X = X_0 + B + X^c + \kappa \star (\mu - \nu) + \kappa' \star \mu. \tag{2.4}$$

Here and below we use standard notation for stochastic integrals and characteristics, see for example [6] for all unexplained notation.

We are interested in the associated processes $V^n(f)$ and $V'^n(f)$ in (1.2) (written as $V^n(f;X)$ and $V'^n(f;X)$ if we want to emphasize the dependency upon X). For simplicity we write $\mathbb{P}_i^n$ and $\mathbb{E}_i^n$ for the conditional probability and expectation w.r.t. $\mathcal{F}_{i\Delta_n}$. We also introduce some related notation, where f is a small enough function (e.g. bounded) :

$$H_i^n(f) = \mathbb{E}_{i-1}^n(f(\Delta_i^n X)), \qquad K_i^n(f) = \mathbb{E}_{i-1}^n(f(\Delta_i^n X/\sqrt{\Delta_n})), \tag{2.5}$$

$$\overline{H}^n(f)_t := \sum_{i=1}^{[t/\Delta_n]} H_i^n(f), \qquad \overline{K}^n(f)_t := \sum_{i=1}^{[t/\Delta_n]} K_i^n(f). \tag{2.6}$$

Our first key result needs no special assumption, but stating it requires some additional notation: first, $C^{0,\nu}$ denotes the set of all functions on $\mathbb{R}$ which are $\nu(\omega;\mathbb{R}_+\times\ dx)$–a.e. continuous, for $\mathbb{P}$–almost all ω. Next, we set

$$I=\{r\geq 0:\phi_r\star\nu_t<\infty\quad\forall t>0\}. \tag{2.7}$$

This is an interval of the form $[\alpha,\infty)$ or (α,∞), for some $\alpha\in[0,2]$. We have $2\in I$ always, and we have $X-X^c\in\mathcal{V}$ if and only if $1\in I$, and X has a.s. finitely many jumps on each finite time interval if and only if $0\in I$. Set

$$\left.\begin{array}{l} X'=X-X^c-X_0, \\ 1\in I\quad\Rightarrow\quad \overline{B}=B-\kappa\star\nu,\quad X''_t=\sum_{s\leq t}\Delta X_s. \end{array}\right\} \tag{2.8}$$

So if $1\in I$ we have $X'=\overline{B}+X''$, and $\overline{B}$ is the "genuine" drift. In this case $\overline{B}\in\mathcal{V}$.

The other results need various assumptions, which we presently describe.

Hypothesis (H) : The characteristics (B,C,ν) of X have the form

$$B_t=\int_0^t b_s ds,\qquad C_t=\int_0^t c_s ds,\qquad \nu(dt,dx)=dt\ F_t(dx). \tag{2.9}$$

Moreover the processes (b_t) and $(F_t(\phi_2))$ are locally bounded predictable (where $F_t(f)=\int f(x)F_t(dx)$), and the process (c_t) is càdlàg adapted. $\square$

Clearly (H) implies the quasi-left-continuity of X. Under (H), we write

$$\sigma_t\ =\ \sqrt{c_t}. \tag{2.10}$$

As is well known, the form (2.9) of the characteristics of X is equivalent to the fact that X can be written as

$$X_t=X_0+\int_0^t b_s ds+\int_0^t \sigma_s dW_s+\kappa(\delta)\star(\underline{\mu}-\underline{\nu})_t+\kappa'(\delta)\star\underline{\mu}_t, \tag{2.11}$$

where

1) σ is given by (2.10) and δ is a "predictable" map from $\Omega\times\mathbb{R}_+\times\mathbb{R}$ on $\mathbb{R}$, connected with F_t by the fact that $F_t(\omega,dx)$ is the image of the Lebesgue measure on $\mathbb{R}$ by the map $x\mapsto\delta(\omega,t,x)$.

2) W and $\underline{\mu}$ are a Wiener process and a Poisson random measure on $\mathbb{R}_+\times\mathbb{R}$ on the filtered space $(\Omega,\overline{\mathcal{F}},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ and the predictable compensator of $\underline{\mu}$ is $\underline{\nu}(ds,dx)=ds\otimes dx$ (we may have to enlarge a bit the original space in order to accommodate the pair $(W,\underline{\mu})$).

Hypothesis (K) : (H) holds and in (2.11) the coefficient δ satisfies $|\delta(\omega,t,x)|\leq\gamma_k(x)$ for all $t\leq T_k(\omega)$, where γ_k are (deterministic) functions on $\mathbb{R}$ with $\int\phi_2\circ\gamma_k(x)\ dx<\infty$, and (T_k) is a sequence of stopping times increasing to $+\infty$. $\square$

In the next hypothesis we assume that the space also supports a second Wiener process W' independent of W. Note that the particular form of $\underline{\mu}$ in (2.11) or in (2.12) below is actually irrelevant, it could be a Poisson random measure on $\mathbb{R}_+ \times E$ for any space E and with a compensator of the form $dt \otimes \underline{F}(dx)$, provided the measure $\underline{F}$ is infinite and without atom; or, we could have two different Poisson random measure in (2.11) and in (2.12).

Hypothesis (L-s) : (H) holds and the process σ in the formula (2.11) has the form

$$\sigma_t = \sigma_0 + \int_0^t \widetilde{b}_s ds + \int_0^t \widetilde{\sigma}_s dW_s + \int_0^t \widetilde{\sigma}'_s dW'_s + \kappa(\widetilde{\delta}) \star (\underline{\mu} - \underline{\nu})_t + \kappa'(\widetilde{\delta}) \star \underline{\mu}_t, \tag{2.12}$$

and

a) the process $(\widetilde{b}_t)$ is optional and locally bounded;

b) the processes (b_t), $(\widetilde{\sigma}_t)$, $(\widetilde{\sigma}'_t)$ are adapted left-continuous with right limits and locally bounded;

c) the functions $\delta(\omega, t, x)$ and $\widetilde{\delta}(\omega, t, x)$ are predictable, left-continuous with right limits in t, and $|\delta(\omega, t, x)| \leq \gamma_k(x)$ and $|\widetilde{\delta}(\omega, t, x)| \leq \widetilde{\gamma}_k(x)$ for all $t \leq T_k(\omega)$, where $\gamma_k, \widehat{\gamma}_k$ are (deterministic) functions on $\mathbb{R}$ with $\int \phi_s \circ \gamma_k(x)\, dx < \infty$ (with $0^0 = 0$) and $\int \phi_2 \circ \widetilde{\gamma}_k(x)\, dx < \infty$, and (T_k) is a sequence of stopping times increasing to $+\infty$. □

In (L-s) we implicitly assume $s \in [0, 2]$. Note that if $s \leq s' \leq 2$, then (L-s') $\Rightarrow$ (L-s) $\Rightarrow$ (K) $\Rightarrow$ (H), and (L-s) implies that $s \in I$. It is worthwhile to emphasize that (L-0) implies that X has locally finitely many jumps, and also that when X is continuous then *all hypotheses* (L-s) for $s \in [0, 2]$ are identical.

Finally we have an assumption of a different nature :

Hypothesis (H') : We have (H) and the processes (c_t) and (c_{t-}) do not vanish. □

Remark 2.1 These assumptions, and especially (L-s), may appear complicated to check. However, if X is one of the components of the solution of an SDE of the form $d\overline{X}_t = f(\overline{X}_{t-})dZ_t$, where Z is a multidimensional Lévy process and f is a C^2 function with linear growth and locally bounded second derivative, then (L-2) is automatically satisfied. The same holds for solutions of SDEs driven by W and $\underline{\mu}$. □

2.3 The laws of large numbers.

First, we have a result valid with no assumption at all on X (recall the notation (1.2), (2.7) and (2.8)) :

Theorem 2.2 *(i) The processes $V^n(f)$ converge in probability in the Skorokhod sense to a suitable limit $V(f)$ in the following cases:*

(a) With $V(f) = f \star \mu$, when

[a-1] $f \in \mathcal{E}_2''' \cap C^{0,\nu}$,

[a-2] $f \in \mathcal{E}_r'' \cap C^{0,\nu}$ *if* $r \in I \cap (1,2)$ *and* $C = 0$,

[a-3] $f \in \mathcal{E}_1''' \cap C^{0,\nu}$ *if* $1 \in I$ *and* $C = 0$,

[a-4] $f \in \mathcal{E}_r'' \cap C^{0,\nu}$ *if* $r \in I \cap (0,1]$ *and* $C = \overline{B} = 0$.

(b) With $V(f) = f \star \mu + C$, *when* $f \in \mathcal{E}_2' \cap C^{0,\nu}$.

(c) With $V(f) = f \star \mu + v(\overline{B})$, *when* $f \in \mathcal{E}_1' \cap C^{0,\nu}$ *and* $C = 0$ *and* $1 \in I$.

(ii) Moreover in (a) and (c) above we also have $V^n(f) - V(f)^{(n)} \xrightarrow{v.p.} 0$.

When $f = h_r$ the case (b) ($r = 2$) is well known (convergence of the realized quadratic variation), and (a) for $r > 2$ may be found in [10] for general semimartingales, and (c) ($r = 1$) is also well known because $V(f)$ is then the variation process of X.

The next LLNs are obtained after centering or normalization. For the first one we need to introduce the process

$$\Sigma(f, \psi_\eta) = (f\psi_\eta) \star (\mu - \nu) + (f(1 - \psi_\eta) \star \mu, \tag{2.13}$$

which is well defined for $\eta \in (0, \infty]$ as soon as $f^2 \in \mathcal{E}_r''$ for some $r \in I$, and also for $\eta = \infty$ if further f is bounded (it is then a locally square-integrable martingale).

Theorem 2.3 *Assume that* X *is quasi-left-continuous. Let* $f \in \mathcal{E}_r'' \cap C^{0,\nu}$ *for some* $r \in (1,2)$. *Then* $V^n(f) - \overline{H}^n(f\psi_\eta) \xrightarrow{Sk.p.} \Sigma(f, \psi_\eta)$ *if* $\eta < \infty$, *and also if* $\eta = \infty$ *when* f *is bounded.*

Theorem 2.4 *Assume (H). Then:*

(i) $\Delta_n V'^n(g)_t \xrightarrow{u.c.p.} \int_0^t \rho_{\sigma_u}(g)du$ *if* g *is a continuous function, in* $\mathcal{E}$ *when* X *is continuous, and with* $g(x)/x^2 \to 0$ *as* $|x| \to \infty$ *otherwise.*

(ii) $\Delta_n^{1-r/2} V^n(f)_t \xrightarrow{u.c.p.} m_r \int_0^t c_u^{r/2} du$ *if* $f \in \mathcal{E}_r'$ *and* $r \in (0,2)$.

(iii) $V''^n(\varpi, \alpha) \xrightarrow{u.c.p.} C_t$ *for all* $\varpi \in (0, \frac{1}{2})$ *and* $\alpha > 0$.

Remark 2.5 Theorem 2.3 is an LLN because the convergence holds in probability, but it can also be viewed as a CLT since the limiting process is a (local) martingale as soon as f is bounded and $\eta = \infty$. $\square$

Remark 2.6 Theorem 2.3 overlaps with (i) of Theorem 2.2, but in the overlapping cases the two are of course consistent. When Theorem 2.3 applies and Theorem 2.2 fails, there is $t > 0$ such that both sequences $(V^n(f)_t)$ and $(\overline{H}^n(f\varphi)_t)$ are *not* tight.

When $r \in (1,2)$ and $f \in \mathcal{E}_r' \cap C^{0,\nu}$, Theorems 2.3 and 2.4-(ii) also overlap: an equivalent way of writing the later is $\Delta_n^{1-r/2} \left(V^n(f) - \overline{H}^n(f)\right) \xrightarrow{\text{u.c.p.}} 0$ (see the proofs below), so Theorem 2.3 in this case is the CLT associated with the LLN of Theorem 2.4-(ii) in a sense, or perhaps rather as a "second order" LLN because the convergence takes place in probability. $\square$

Remark 2.7 The reader will note the - different - assumptions in the last two theorems. Theorem 2.3 probably fails if X is not quasi-left continuous. Theorem 2.4 just makes no sense if (H) fails (or rather, if the second equality in (2.9) fails), and the quasi-left continuity is by no means enough for it. $\square$

2.4 The central limit theorems.

The various CLTs below involve stable convergence in law, for which we need some ingredients. Consider an auxiliary space $(\Omega', \mathcal{F}', \mathbb{P}')$ supporting a d-dimensional Brownian motion $\overline{W} = (\overline{W}^j)_{1\leq j\leq d}$, two sequences (U_n) and (U'_n) of $\mathcal{N}(0,1)$ variables, and a sequence (κ_n) of variables uniformly distributed on $(0,1)$, all of these being mutually independent. Then put

$$\widetilde{\Omega} = \Omega \times \Omega', \qquad \widetilde{\mathcal{F}} = \mathcal{F} \otimes \mathcal{F}', \qquad \widetilde{\mathbb{P}} = \mathbb{P} \otimes \mathbb{P}'. \tag{2.14}$$

and extend the variables X_t, b_t, ... defined on Ω and $\overline{W}$, U_n,... defined on Ω' to the product $\widetilde{\Omega}$ in the obvious way, without changing the notation. We write $\widetilde{\mathbb{E}}$ for the expectation w.r.t. $\widetilde{\mathbb{P}}$. Finally, denote by $(T_n)_{n\geq 1}$ an enumeration of the jump times of X which are stopping times, and let $(\widetilde{\mathcal{F}}_t)$ be the smallest (right-continuous) filtration of $\widetilde{\mathcal{F}}$ containing the filtration $(\mathcal{F}_t)$ and w.r.t. which $\overline{W}$ is adapted and such that U_n and U'_n and κ_n are $\widetilde{\mathcal{F}}_{T_n}$-measurable for all n.

Obviously, $\overline{W}$ is an $(\widetilde{\mathcal{F}}_t)$-Brownian motion under $\widetilde{\mathbb{P}}$, as well as W, and W' under (L-2), whereas $\underline{\mu}$ is still a Poisson measure with compensator $\underline{\nu}$ for this bigger filtration. The dimension d of $\overline{W}$ is the number of processes for which we want to have a joint CLT in Theorem 2.16 below, in the other theorems we have $d = 1$ and we then write $\overline{W}^1 = \overline{W}$.

The limiting processes we obtain below are of the form $Y = (Y^j)_{1\leq j\leq d}$ with $Y^j_t = \sum_{k=1}^d \int_0^t \theta^{jk}_u \, d\overline{W}^k_u$ for suitable $(\mathcal{F}_t)$-adapted $d\times d$-dimensional càdlàg processes (θ_t), or the sum of Y_t plus a process of the form

$$Z(g)_t = \sum_{p:\ T_p\leq t} g(\Delta X_{T_p})\Big(\sqrt{\kappa_p}\ U_p\ \sigma_{T_p-} + \sqrt{1-\kappa_p}\ U'_p\ \sigma_{T_p}\Big), \tag{2.15}$$

for some function $g \in \mathcal{E}''_1$. As we will check in Lemma 5.10 below, this formula defines a semimartingale on the extended space, whose conditional law w.r.t. $\mathcal{F}$ depends on the processes X and c (or σ) but not on the particular choice of the stopping times T_n. Moreover, again conditionally on $\mathcal{F}$, the two processes Y and $Z(g)$ are independent and are *martingales* with variance-covariance given by

$$\left.\begin{array}{rcl} \widetilde{\mathbb{E}}(Y^j_t Y^k_t \mid \mathcal{F}) & = & \int (\theta_u\theta^\star_u)^{jk}\ du \\ \widetilde{\mathbb{E}}(Z(g)^2_t \mid \mathcal{F}) & = & C(g)_t \ := \ \sum_{p:\ T_p\leq t} g(\Delta X_{T_p})^2 (c_{T_p-} + \frac{1}{2}\ \Delta c_{T_p}), \end{array}\right\} \tag{2.16}$$

where $\theta^\star$ is the transpose and Δc_{T_p} is the jump of the process (c_t) at time T_p. Moreover, conditionally on $\mathcal{F}$, Y is even a *Gaussian martingale*, and $Z(g)$ also as soon as the processes X and σ have no common jumps. This will also be checked later.

Now we state a variety of CLTs, related with some of the LLNs given above, although the picture is far from being complete. As said before, Theorem 2.3 is already a CLT in a sense, and we start with a result extending this theorem to the case $r = 1$. The other CLTs are related to Theorem 2.4 and with a special case of Theorem 2.2-(a), with unfortunately some unwanted restrictions. We complement these CLTs with some "tightness" results, in view of applications. Finally we will end up with a multidimensional CLT which contains the previous results and is complicated to state, but which probably is the most useful result for practical applications, at least for those we have in mind.

Theorem 2.8 *Assume (H). Let $f \in \mathcal{E}'_1 \cap C^{0,\nu}$ and $\eta < \infty$, or $\eta = \infty$ if f is bounded. Then* $V^n(f) - \overline{H}^n(f\psi_\eta) \stackrel{\mathcal{L}-(s)}{\longrightarrow} \Sigma(f,\psi_\eta)_t + \sqrt{m_2 - m_1^2}\int_0^t \sigma_u \, d\overline{W}_u$ (note that $m_2 - m_1^2 = 1 - 2/\pi$).

Theorem 2.9 *Assume (L-s), and let g be an even C_b^2 function on $\mathbb{R}$.*

(i) $\frac{1}{\sqrt{\Delta_n}}\left(\Delta_n V'^n(g)_t - \int_0^t \rho_{\sigma_u}(g)du\right) \stackrel{\mathcal{L}-(s)}{\longrightarrow} \int_0^t \sqrt{\rho_{\sigma_u}(g^2) - (\rho_{\sigma_u}(g))^2}\, d\overline{W}_u$ *if* $s \leq 1$;

(ii) $\Delta_n V'^n(g)_t - \int_0^t \rho_{\sigma_u}(g)du = \mathrm{o}_{Pu}(\Delta_n^{1-s/2})$ *otherwise.*

When X is continuous, we have (i) under (L-2), as soon as g is C^1 and even and $g' \in \mathcal{E}$.

Theorem 2.10 *Assume (L-s) and (H'), and let $f \in \mathcal{E}_r$ for some $r \in (0,1]$.*

(i) $\frac{1}{\sqrt{\Delta_n}}\left(\Delta_n^{1-r/2} V^n(f)_t - m_r \int_0^t c_u^{r/2}\right) \stackrel{\mathcal{L}-(s)}{\longrightarrow} \sqrt{m_{2r} - m_r^2}\int_0^t c_u^{r/2}\, d\overline{W}_u$ *if either* $s \leq \frac{2}{3}$ *and* $r < 1$, *or* $\frac{2}{3} < s < 1$ *and* $\frac{1-\sqrt{3s^2-8s+5}}{2-s} < r < 1$;

(ii) $\Delta_n^{1-r/2} V^n(f)_t - m_r \int_0^t c_u^{r/2} = \mathrm{o}_{Pu}\left(\Delta_n^{\frac{(2-s)(1+r)(2-r)}{4+2s(1-r)} - \varepsilon}\right)$ *for all* $\varepsilon > 0$, *otherwise.*

When X is continuous, we have (i) under (L-2) and (H') when $r \in (0,1]$, and also under (L-2) only when $r > 1$.

Theorem 2.11 *Assume (L-s), and let $\varpi \in (0, \frac{1}{2})$ and $\alpha > 0$. Then*

(i) $\frac{1}{\sqrt{\Delta_n}}(V''^n(\varpi,\alpha)_t - C_t) \stackrel{\mathcal{L}-(s)}{\longrightarrow} \sqrt{2}\int_0^t c_u \, d\overline{W}_u$ *if* $s \leq \frac{4\varpi - 1}{2\varpi}$ *(hence* $\varpi \geq \frac{1}{4}$ *and* $s < 1$*);*

(ii) $V''^n(\varpi,\alpha)_t - C_t = \mathrm{o}_{Pu}(\Delta_n^{(2-s)\varpi})$ *otherwise.*

Theorem 2.12 *Let f be a C^1 function on $\mathbb{R}$.*

(i) Under (K), and if f is C^2 on a neighborhood of 0, with $f(0) = f'(0) = 0$ and $f''(x) = o(|x|)$ as $x \to 0$, then $\frac{1}{\sqrt{\Delta_n}}(V^n(f)_t - V(f)_t^{(n)}) \stackrel{\mathcal{L}-(s)}{\longrightarrow} Z(f')_t$ *(with $V(f) = f \star \mu$).*

(ii) Under (L-2) and if $f \in \mathcal{E}_2$ we have $\frac{1}{\sqrt{\Delta_n}}(V^n(f)_t - V(f)_t^{(n)}) \stackrel{\mathcal{L}-(s)}{\longrightarrow} Z(f')_t + \sqrt{2}\int_0^t c_u \, d\overline{W}_u$ *(with $V(f) = C + f \star \mu$).*

Remark 2.13 We do not have stable convergence in law of the processes $\frac{1}{\sqrt{\Delta_n}}(V^n(f) - V(f)_t)$ in the last theorem, and not even mere convergence in law, because of some peculiarity of the Skorokhod topology. However these processes converge finite-dimensionally stably in law to the limits described above. □

Remark 2.14 The limiting process in (ii) of Theorem 2.12 looks pretty much like the limiting process obtained in [7] for the error term in the Euler approximation of the solution of SDEs driven by Lévy processes. This is of course not just by chance ! □

Remark 2.15 Suppose that $f = h_r$. We have a CLT for $V^n(f)$ in the following cases :

• if $r < 1$, in Theorem 2.9 (subject to some - perhaps unnecessary - restrictions on the value of s for which (L-s) holds), after normalization and centering;

• if $r = 1$, in Theorem 2.8, after centering;

• if $1 < r < 2$, in Theorem 2.3, after centering;

• if $r = 2$ or $r > 3$, in Theorem 2.12, after normalization and centering.

When $2 < r \le 3$, there is no CLT, at least with the natural centering of the associated LLN, although a CLT with a more adequate centering might exist: see [8] for a more thorough description of this fact when X is a Lévy process. □

Finally, we give the announced multidimensional CLT, in which we consider components as in Theorems 2.9, 2.10, 2.11 and 2.12. Below we have a d–dimensional process and the index set $\{1, \ldots, d\}$ for the components is partitioned into five (possibly empty) subsets J_l. We consider the process $Y^n = (Y^{n,j})_{1 \le j \le d}$ having the following components :

•$j \in J_1 \quad \Rightarrow \quad Y_t^{n,j} = \Delta_n V'^n(f_j)_t - \int_0^t \rho_{\sigma_u}(f_j)du$, where f_j is C_b^2 and even;

•$j \in J_2 \quad \Rightarrow \quad Y_t^{n,j} = \Delta_n^{1-r(j)/2} V^n(f_j)_t - m_{r(j)} \int_0^t \sigma_u^{r(j)/2} du$, where $f_j \in \mathcal{E}_{r(j)}$ for some $r(j) \in (0,1)$ in general or $r(j) \in (0,\infty)$ if X is continuous;

•$j \in J_3 \quad \Rightarrow \quad Y_t^{n,j} = V''^n(\varpi_j, \alpha_j)_t - C_t$, where $\varphi_j \in [1/4, 1/2)$ and $\alpha_j > 0$; we then put $r(j) = 2$;

•$j \in J_4 \quad \Rightarrow \quad Y^{n,j} = V^n(f_j) - V(f_j)^{(n)}$, where f_j is C^1, and C^2 on a neighborhood of 0 with $f_j(0) = f'_j(0) = 0$ and $f''_j(x) =$o$(|x|)$ as $x \to 0$;

•$j \in J_5 \quad \Rightarrow \quad Y^{n,j} = V^n(f_j) - V(f_j)^{(n)}$, where $f_j \in \mathcal{E}_2 \cap C^1$; we then put $r(j) = 2$.

Theorem 2.16 *With the previous setting, we assume (H') if $J_2 \neq \emptyset$, and (L-s) for some $s \in [0,2]$ satisfying*

$J_1 \neq \emptyset \quad \Rightarrow \quad s < 1,$

$J_2 \neq \emptyset \quad \Rightarrow \quad$ *either* $0 \le s \le \frac{2}{3}$ *or* $\frac{2}{3} < s < 1$ *and* $\frac{1-\sqrt{3s^2-8s+5}}{2-s} < \inf_{j \in J_2} r(j),$

$J_3 \neq \emptyset \quad \Rightarrow \quad s < \inf_{j \in J_3} \frac{4\varpi_j - 1}{2\varpi_j}.$

Then $\frac{1}{\sqrt{\Delta_n}} Y^n \stackrel{\mathcal{L}-(s)}{\longrightarrow} Y$, *where*

$$Y_t^j = \begin{cases} \sum_{k \in J_1 \cup J_2 \cup J_3 \cup J_5} \int_0^t \theta_u^{jk} \, d\overline{W}_u^k & \text{if } j \in J_1 \cup J_2 \cup J_3 \\ Z(f'_j)_t & \text{if } j \in J_4, \\ Z(f'_j)_t + \sum_{k \in J_1 \cup J_2 \cup J_3 \cup J_5} \int_0^t \theta_u^{jk} \, d\overline{W}_u^k & \text{if } j \in J_5, \end{cases} \tag{2.17}$$

and where $\theta = (\theta^{jk})_{j,k\in J_1\cup J_2\cup J_4}$ is an $(\mathcal{F}_t)$-adapted càdlàg process whose square $\theta\theta^\star$ is the symmetric matrix characterized by

$$(\theta_t\theta_t^\star)^{jk} = \begin{cases} \rho_{\sigma_t}(f_j f_k) - \rho_{\sigma_t}(f_j)\rho_{\sigma_t}(f_k), & j,k \in J_1 \\ (m_{r(j)+r(k)} - m_{r(j)}m_{r(k)})c_t^{r(j)/2+r(k)/2}, & j,k \in J_2\cup J_3\cup J_5 \\ \rho_{\sigma_t}(h_{r(j)}f_k) - \rho_{\sigma_t}(h_{r(j)})\rho_{\sigma_t}(f_k), & j\in J_2\cup J_3\cup J_5,\ k\in J_1. \end{cases} \tag{2.18}$$

When X is continuous, the same holds under (L-2) and (H') as soon as the f_j's for $j\in J_1$ are C^1 and even with $f'_j \in \mathcal{E}$, and $r(j)\in(0,\infty)$ for $j\in J_2$, and one can relax (H') if $r(j)>1$ for all $j\in J_2$.

(It is easy to check that the right side of (2.18) is a positive symmetric matrix indexed by $J_1\cup J_2\cup J_3\cup J_5$, so it has a "square-root" θ_t).

3 Theorems 2.2 and 2.3

3.1 Proof of Theorem 2.2.

The idea of the proof is the same as in [8], but the details are slightly more involved, so we give a complete proof.

Step 1. If f satisfies any one of the conditions in (i) the process $f\star\nu$ is in $\mathcal{V}$, hence $V(f)=f\star\mu$ as well. In view of the convergence $V(f)^{(n)} \xrightarrow{\text{Sk}} V(f)$ it is clear that (ii) implies (i-a) and (i-c). Below, we use the notation

$$Z^n(f) = V^n(f) - V(f)^{(n)}. \tag{3.1}$$

Step 2: Here we prove (i) and (ii) when $f\in C^{0,\nu}$ vanishes on a neighborhood of 0, say $[-2\varepsilon,2\varepsilon]$, hence $V(f)=f\star\mu$. For any fixed $\varepsilon>0$ we set :

$$\left.\begin{array}{l} \bullet\ S_1, S_2,\cdots \text{ are the successive jump times of } X \text{ with } |\Delta X_t|>\varepsilon, \\ \bullet\ R_p = \Delta X_{S_p}, \\ \bullet\ X(\varepsilon)_t = X_t - (x1_{\{|x|>\varepsilon\}})*\mu_t = X_t - \sum_{p:\ S_p\le t} R_p, \\ \bullet\ R'^n_p = \Delta_i^n X(\varepsilon) \text{ on the set } \{(i-1)\Delta_n < S_p \le i\Delta_n\}, \\ \bullet\ \Omega_n(T,\varepsilon) \text{ is the set of all } \omega \text{ such that each interval } [0,T]\cap((i-1)\Delta_n, i\Delta_n] \\ \quad \text{contains at most one } S_p(\omega), \text{ and that } |\Delta_i^n X(\varepsilon)(\omega)|\le 2\varepsilon \text{ for all } i\le T/\Delta_n. \end{array}\right\} \tag{3.2}$$

All these depend on ε of course, and $\Omega_n(T,\varepsilon)\to\Omega$ as $n\to\infty$.

Recalling $f(x)=0$ when $|x|\le 2\varepsilon$, we see that on the set $\Omega_n(T,\varepsilon)$ and for all $t\le T$,

$$v(Z^n(f))_t = \sum_{p:\ S_p\le \Delta_n[t/\Delta_n]} |(f(R_p+R'^n_p) - f(R_p))|, \tag{3.3}$$

Since $f\in C^{0,\nu}$ there is a null set N such that, if $\omega\notin N$, then f is continuous at each point $R_p(\omega)$, whereas $R'^n_p(\omega)\to 0$, so $v(Z^n(f))_T\to 0$ when $\omega\notin N$. Hence (ii) is obvious (we even have almost sure convergence).

Step 3: Here we prove (ii) in case (c), so we assume $1 \in D$ and $C = 0$. As said before, $X \in \mathcal{V}$ and $v(X - X_0) = V(h_1)$ (recall (2.2)), and it is well known that $V^n(h_1)_t$ converges pointwise to $V(h_1)_t$. Then $Z^n(h_1)_t \to 0$ and, since $Z^n(h_1)$ is a nonpositive decreasing process, we in fact have $v(Z^n(h_1))_t \to 0$ for all t.

Now let $f \in \mathcal{E}'_1 \cap C^{0,\nu}$. We have $|(f - h_1)\psi_\eta| \leq \varepsilon_\eta h_1$, where $\varepsilon_\eta \to 0$ as $\eta \to 0$. We have

$$v(Z^n(f)) \;\leq\; (1 + \varepsilon_\eta) v(Z^n(h_1)) + v(Z^n((f - h_1)(1 - \psi_\eta))) + \varepsilon_\eta V(h_1).$$

The first two terms on the right go to 0 a.s. by the above and Step 2, and $V(h_1)$ is finite-valued and $\varepsilon_\eta \to 0$, hence the result.

Step 4: Here we prove the remaining claims (ii-a) and (i–b), assuming that

$$Z^n(h_r\psi_\eta) \;\stackrel{\text{u.c.p.}}{\longrightarrow}\; 0 \tag{3.4}$$

in the relevant cases: that is either $r = 2$ (hence $V(h_2\psi_\eta) = C + (h_2\psi_\eta) \star \mu$), or $r \in I \cap (1, 2)$ and $C = 0$, or $r \in I \cap (0, 1)$ and $C = \overline{B} = 0$ (so $V(h_2\psi_\eta) = (h_2\psi_\eta) \star \mu$ in these two cases).

Assume $f \in \mathcal{E}'''_2 \cap C^{0,\nu}$. Then $|f\psi_\eta| \leq \varepsilon_\eta h_2\psi_\eta$, with $\varepsilon_\eta \to 0$ as $\eta \to 0$, and thus

$$v(Z^n(f)) \leq v(Z^n(f(1 - \psi_\eta)) + \varepsilon_\eta(|Z^n(h_2\psi_\eta)| + 2V(h_2\psi_\eta)).$$

The first term on the right goes to 0 a.s. by Step 2, so (3.4) and $\varepsilon_\eta \to 0$ and $V(h_2\psi_\eta)_t < \infty$ give (ii) in case [a-1]. When $f \in \mathcal{E}'''_1 \cap C^{0,\nu}$ and $1 \in I$ and $C = 0$, the same argument with h_1 instead of h_2 works (use Step 3), and we have (ii) in case [a-3].

When $f \in \mathcal{E}''_r \cap C^{0,\nu}$ with $r < 2$, we have $|f\psi_\eta| \leq K h_r\psi_\eta$ for all η small enough, hence

$$v(Z^n(f)) \leq v(Z^n(f(1 - \psi_\eta))) + K|Z^n(f_r\psi_\eta)| + 2K(f_r\psi_\eta) \star \mu.$$

The first two terms on the right go to 0 in probability by Step 2 and (3.4), and the third term goes to 0 as $\eta \to 0$ because $r \in I$. So we have (ii) in cases [a-2] and [a-4].

Finally let $f \in \mathcal{E}'_2 \cap C^{0,\nu}$, so $|(f - h_2)\psi_\eta| \leq \varepsilon_\eta h_2\psi_\eta$, with $\varepsilon_\eta \to 0$ as $\eta \to 0$, and thus

$$|Z^n(f)| \;\leq\; v(Z^n(f(1 - \psi_\eta))) + (1 + \varepsilon_\eta)|Z^n(h_2\psi_\eta)| + \varepsilon_\eta + V(h_2),$$

and we conclude (i–b) as above.

Step 5: We are left to prove (3.4). In other words, it is enough to prove that if f is C^2 outside 0, with compact support and $f(x) = |x|^r$ around 0, and when either $r = 2$, or $1 < r < 2$ and $C = 0$, or $0 < r < 1$ and $C = \overline{B} = 0$, then we have $Z^n(f) \stackrel{\text{u.c.p.}}{\longrightarrow} 0$. Set

$$g(x, y) = f(x + y) - f(x) - f(y) - \kappa(x)f'(y), \quad k(x, y) = f(x + y) - f(x) - f(y)$$

with the convention $f'(0) = 0$ if $r < 1$ (otherwise, $f'(0)$ is the derivative of f at 0, of course). Recall that $V(f) = f \star \mu$ when $r < 2$ and $V(f) = C + f \star \mu$ if $r = 2$.

Define the process Y^n by $Y^n_t = X_t - X_{(i-1)\Delta_n}$ for $t \in [(i - 1)\Delta_n, i\Delta_n]$. Itô's formula when $r = 2$ and its extension as given in Theorem 3.1 of [9] when $r < 2$ give us

$$Z^n(f)_t = \sum_{i=1}^{[t/\Delta_n]} \big(f(Y^n_{i\Delta_n}) - \Delta^n_i V(f)\big) = \sum_{i=1}^{[t/\Delta_n]} (A^n_i + M^n_i),$$

where (recall that $C = X^c = 0$ when $r < 2$ here, so f'' does not occur below in that case)

$$\left.\begin{array}{l} A_i^n = \begin{cases} \int_{(i-1)\Delta_n}^{i\Delta_n} \big(f'(Y_{s-}^n)dB_s + (\frac{1}{2}\ f''(Y_s^n) - 1)dC_s\big) & \\ \qquad\qquad + \int_{(i-1)\Delta_n}^{i\Delta_n} \int_{\mathbb{R}} g(x, Y_{s-}^n)\nu(ds, dx) & \text{if } 1 < r \le 2 \\ \int_{(i-1)\Delta_n}^{i\Delta_n} \int_{\mathbb{R}} k(x, Y_{s-}^n)\nu(ds, dx) & \text{if } 0 < r < 1, \end{cases} \\ \\ M_i^n = \int_{(i-1)\Delta_n}^{i\Delta_n} f'(Y_{s-}^n)dX_s^c + \int_{(i-1)\Delta_n}^{i\Delta_n} \int_{\mathbb{R}} k(x, Y_{s-}^n)(\mu - \nu)(ds, dx). \end{array}\right\} \tag{3.5}$$

In other words, $Z^n(f) = A(n)^{(n)} + M(n)^{(n)}$, where

$$A(n)_t = \begin{cases} \int_0^t \big(f'(Y_{s-}^n)dB_s + (\frac{1}{2}\ f''(Y_-^n) - 1)dC_s\big) + g(x, Y_-^n) \star \nu_t & \text{if } 1 < r \le 2 \\ k(x, Y_-^n) \star \nu_t & \text{if } 0 < r < 1, \end{cases} \tag{3.6}$$

$$M(n)_t = \int_0^t f'(Y_{s-}^n)dX_s^c + k(x, Y_-^n) \star (\mu - \nu)_t.$$

In particular $M(n)$ is a locally square–integrable martingale, whose predictable bracket $\langle M(n), M(n)\rangle$ is such that $A'(n) - \langle M(n), M(n)\rangle$ is non–decreasing (see Theorem II.1.33 of [6]), where

$$A'(n) = f'(Y^n)^2 \bullet C + k(x, Y_-^n)^2 \star \nu, \tag{3.7}$$

Step 6: At this stage, it remains to prove that $A(n) \stackrel{\text{u.c.p.}}{\longrightarrow} 0$ and $M(n) \stackrel{\text{u.c.p.}}{\longrightarrow} 0$, and for the last property Lenglart domination property (Lemma I.3.30 of [6]) it is enough to prove $A'(n) \stackrel{\text{u.c.p.}}{\longrightarrow} 0$.

Suppose first that $1 < r \le 2$, so the function f is C_b^1 and f' is Hölder with index $r-1$, and $f(0) = f'(0) = 0$, hence $|k(x,y)| \le C\phi_1(x)$ and $|g(x,y)| \le C\phi_r(x)$, and obviously $f'(y)$ and $k(x,y)$ and $g(x,y)$ all go to 0 as $y \to 0$ Moreover if $r = 2$ we also have $\frac{1}{2}f''(y) - 1 \to 0$ as well. By the assumption that $r \in I$ we have $\phi_r \star \nu_t < \infty$, and a fortiori $\phi_1^2 \star \nu_t < \infty$, for all $t > 0$. Since $Y_{s-}^n \to 0$ as $n \to \infty$, we deduce from the dominated convergence theorem and also from the property $C = 0$ when $r < 2$ that $\sup_{s \le t} |A(n)_s|$ and $\sup_{s \le t} A'(n)_s$ both go to 0 pointwise, and the result is proved.

Second, assume that $r < 1$. Then $|k(x,y)| \le C\phi_r(x)$, and again $k(x,y) \to 0$ as $y \to 0$. Then we conclude as above. □

3.2 Some consequences.

Now we derive some "technical" consequences of this basic result.

Lemma 3.1 *Suppose that the pair (X, f) satisfies one of the conditions of Theorem 2.2–(i), and also that f is bounded. Then we have:*

(i) $\overline{H}^n(f) = \mathrm{O}_{Pu}(1)$.

(ii) If X is quasi–left continuous, $\overline{H}^n(f) \stackrel{u.c.p.}{\longrightarrow} \overline{H}(f)$, where $\overline{H}(f)$ is the (continuous) predictable compensator of $V(f)$, that is $\overline{H}(f) = f \star \nu$ in case (a), and $\overline{H}(f) = C + f \star \nu$ in case (b), and $\overline{H}(f) = v(\overline{B}) + f \star \nu$ in case (c).

Our conditions imply that $|f| \star \mu$ is finite with bounded jumps, so$f \star \nu$ is well defined. We cannot hope for (ii) to be true if X is not quasi–left continuous. In general, $\overline{H}^n(f)_t$ goes to $\overline{H}(f)_t$ for any t which is not a fixed time of discontinuity of X, but the convergence is for the weak $\sigma(\mathbb{L}^1, \mathbb{L}^\infty)$ topology on $\mathbb{L}^1$: so it is not likely to be really useful !

Proof. First we observe that if f satisfies the assumptions of case (a) of Theorem 2.2, then the functions f^+, f^- and $|f|$ satisfy the same; when f satisfies the assumptions of cases (b) or (c), then f^+ and $|f|$ satisfy the same, whereas $f^- \in \mathcal{E}_2''' \cap C^{0,\nu}$. So it is enough to prove the result when $f \geq 0$.

Under our assumptions, the increasing processes $V(f)$, $\overline{H}(f)$, $v(B)$, C and $\phi_2 \star \nu$ are locally bounded. So there is a sequence T_p of stopping times increasing to infinity, such that we have identically

$$V(f)_{T_p} + \overline{H}(f)_{T_p} + v(B)_{T_p} + C_{T_p} + \phi_2 \star \nu_{T_p} + \leq K_p. \tag{3.8}$$

Set $\overline{H}^{n,p}(f)_t = \sum_{i=1}^{[t/\Delta_n]} H_i^{n,p}(h)$ and $H_i^{n,p}(f) = \mathbb{E}_{i-1}^n(f(\Delta_i^n X^{T_p}))$. We have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{s \leq t} |\overline{H}^n(f)_s - \overline{H}^{n,p}(f)_s| \; 1_{\{T_p > t\}}\right) \\
&\quad \leq \mathbb{E}\left(\sum_{i=1}^{[t/\Delta_n]} 1_{\{T_p > (i-1)\Delta_n\}} \mathbb{E}_{i-1}^n\left(|f(\Delta_i^n X) - f(\Delta_i^n X^{T_p})|\right)\right) \\
&\quad \leq K\mathbb{E}\left(\sum_{i=1}^{[t/\Delta_n]} 1_{\{T_p > (i-1)\Delta_n\}} \; \mathbb{P}_{i-1}^n(T_p \leq i\Delta_n)\right) \;\; \leq \;\; K\mathbb{P}(T_p \leq t),
\end{aligned}$$

where the second inequality above follows from $0 \leq f \leq K$. Hence we readily deduce the following implications from the fact that $\mathbb{P}(T_p \leq t) \to 0$ as $p \to \infty$ for all t:

$$\left.\begin{array}{lll}
\overline{H}^{n,p}(f)_t = \mathrm{O}_{Pu}(1), \quad \forall p & \Rightarrow & \overline{H}^n(f)_t = \mathrm{O}_{Pu}(1), \\
\overline{H}^{n,p}(f)_t \;\stackrel{\text{u.c.p.}}{\longrightarrow}\; \overline{H}(f)_{t \wedge T_p}, \quad \forall p & \Rightarrow & \overline{H}^n(f)_t \;\stackrel{\text{u.c.p.}}{\longrightarrow}\; \overline{H}(f)_t.
\end{array}\right\} \tag{3.9}$$

Therefore for (i) (resp. (ii)) it is enough to prove the first (resp. second) left side property in (3.9). Equivalently, it is enough to prove the results when X is such that (3.8) holds for $T_1 = \infty$. So we proceed to proving (i) and (ii) under this additional assumption.

(i) Set $S_{n,q} = \inf(t : V^n(f)_t \geq q)$, hence

$$\mathbb{E}(\overline{H}^n(f)_{S_{n,q}}) = \mathbb{E}(V^n(f)_{S_{n,q}}) \leq q + K_1.$$

Now, $V^n(f)_t \stackrel{\mathbb{P}}{\longrightarrow} V(f)_t < \infty$ for all t, hence

$$\lim_{q \to \infty} \; \sup_n \; \mathbb{P}(S_{n,q} < t) = 0. \tag{3.10}$$

Combining these two properties gives the tightness of each sequence $(\overline{H}^n(f)_t)_n$.

(ii) Recall the following property, known as the "approximated Laplacians" property, holds because $V(f)$ is quasi–left continuous, see e.g. [13]):

$$\overline{H}'^n(f)_t := \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}^n_{i-1}(\Delta^n_i V(f)) \quad \stackrel{\text{u.c.p.}}{\longrightarrow} \quad \overline{H}(f)_t \tag{3.11}$$

(in [13] the convergence is for each t, but it is also u.c.p. because both sides are increasing in t, and $\overline{H}(f)$ is continuous).

Now we prove the result in cases (a) and (c) of Theorem 2.2. By (ii) of this theorem we know that $v(V^n(f) - V(f)^{(n)})_t \stackrel{\mathbb{P}}{\longrightarrow} 0$ for all t. By hypothesis $V(f)_\infty \leq K$, so if $S_{n,q}$ is like in (i), we have $\mathbb{E}(v(V^n(f) - V(f)^{(n)})_{t\bigwedge S_{n,q}}) \to 0$, and a fortiori $\mathbb{E}(v(\overline{H}^n(f) - \overline{H}'^n(f))_{t\bigwedge S_{n,q}}) \to 0$. Since (3.10) holds, we deduce the result from (3.11).

Finally we prove the result in case (b). Using the notation of Step 5 of the proof of Theorem 2.2, we have

$$H^n_i(f) = \Delta^n_i \overline{H}^n(f) + \mathbb{E}^n_{i-1}(A^n_i).$$

Then in view of (3.11) it is enough to have $\mathbb{E}(v(A(n)_\infty) \to 0$ (recall (3.6, here $r = 2$). But since $v(B)_\infty$, C_∞, and $\phi_2 \star \nu_\infty$ are bounded, this is proved exactly as $A(n) \stackrel{\text{u.c.p.}}{\longrightarrow} 0$ in Step 6 of the proof of Theorem 2.2. □

Lemma 3.2 *Assume $C = 0$, and let $s \in I \cap [0, 2]$. Let $f \in \mathcal{E}''^b_r$ for some $r > 0$. Then*

$$\overline{H}^n(f) = \begin{cases} \mathrm{o}_{Pu}(\Delta_n^{r/s-1}) & \text{if } r < s,\ s > 1 \\ \mathrm{O}_{Pu}(\Delta_n^{r-1}) & \text{if } r < 1,\ s \leq 1 \\ \mathrm{O}_{Pu}(1) & \text{if } r \geq s \bigvee 1. \end{cases} \tag{3.12}$$

Proof. There is a function $f_r \in \mathcal{E}^b_r \cap C^0$ such that $|f| \leq f_r$. Since $|\overline{H}^n(f)| \leq \overline{H}^n(f_r)$ it is enough to prove the result for f_r. Set $s' = s \bigvee 1$, which is in $I \cap [1, 2]$.

When $r \geq s'$ we have $f_r \in \mathcal{E}'''_2 \cap C^0$ if $r > 2$, and $f_r \in \mathcal{E}_2 \cap C^0$ if $r = 2$, and $f_r \in \mathcal{E}''_{s'} \cap C^0$ if $1 < r < 2$, and $f_r \in \mathcal{E}'_1 \cap C^0$ if $r = 1$, and $r \in I$ always, so f_r is always in one of the cases of Theorem 2.2, and the result follows from Lemma 3.1.

When $r < s'$, Hölder inequality yields for all $\varepsilon > 0$:

$$\Delta_n^{1-r/s'} \overline{H}^n(f_r)_t \leq t^{1-r/s'} \left(\overline{H}^n((f_r\psi_\varepsilon))^{s'/r})_t\right)^{r/s'} + \Delta_n^{1-r/s'} \overline{H}^n(f_r(1-\psi_\varepsilon))_t.$$

Since $f_r(1-\psi_\varepsilon) \in \mathcal{E}'''^b_2 \cap C^0$, by Lemma 3.1 again the last term above goes to 0 in probability for any $\varepsilon > 0$ because $r < s$. Since $(f_r\psi_\varepsilon)^{s'/r} \in \mathcal{E}'^b_{s'} \cap C^0$ we deduce as above, from Lemma 3.1 again, that the first term on the right goes to $t^{1-r/s'}(f_r\psi_\varepsilon)^{s'/r} \star \nu_t$ if $s' > 1$ and to $t^{1-r/s'}\left((f_r\psi_\varepsilon)^{s'/r} \star \nu_t + v(\overline{B})_t\right)$ if $s' = 1$. Now, $(f_r\psi_\varepsilon)^{s'/r} \star \nu_t \to 0$ as $\varepsilon \to 0$ because $s' \in I$. Then we obtain the first and second properties in (3.12). □

3.3 Proof of Theorem 2.3.

Let $f \in \mathcal{E}''_r \cap C^{0,\nu}$ with $r \in (1,2)$. Let $\eta \in (0,\infty)$, or $\eta = \infty$ when f is bounded: in all cases the process $\Sigma(f,\psi_\eta)$ is well defined.

For any $\varepsilon > 0$ we have $f - f\psi_\varepsilon \in \mathcal{E}'''_2 \cap C^{0,\nu}$, so $V^n(f - f\psi_\varepsilon) \stackrel{\text{Sk.p.}}{\longrightarrow} (f - f\psi_\varepsilon) \star \mu$ by Theorem 2.2 and $\overline{H}^n(f\psi_\eta - f\psi_\varepsilon) \stackrel{\text{u.c.p.}}{\longrightarrow} (f(\psi_\eta - \psi_\varepsilon)) \star \nu$ by Lemma 3.1. Therefore, as soon as $\varepsilon < \eta$, $V^n(f(1-\psi_\varepsilon)) - \overline{H}^n(f(1-\psi_\varepsilon)\psi_\eta) \stackrel{\text{Sk.p.}}{\longrightarrow} \Sigma(f(1-\psi_\varepsilon)),\psi_\eta)$. Moreover it is obvious that $\Sigma(f(1-\psi_\varepsilon),\psi_\eta) \stackrel{\text{u.c.p.}}{\longrightarrow} \Sigma(f,\psi_\eta)$ as $\varepsilon \to 0$. Hence in order to prove the result it is enough to show that if $M^n(\varepsilon) = V^n(f\psi_\varepsilon) - \overline{H}^n(f\psi_\varepsilon)$, then we have

$$t > 0,\ \rho > 0 \quad \Rightarrow \quad \lim_{\varepsilon \to 0}\ \limsup_n\ \mathbb{P}(\sup_{s \le t} |M^n(\varepsilon)| > \rho) = 0. \tag{3.13}$$

Now the process $M^n(\varepsilon)$ is a locally bounded martingale w.r.t. the filtration $(\mathcal{F}^n_t = \mathcal{F}_{\Delta_n[t/\Delta_n]})_{t \ge 0}$, and its predictable quadratic variation is

$$C^n(\varepsilon)_t = \sum_{i=1}^{[t/\Delta_n]} \big(H^n_i(f\psi_\varepsilon)^2) - (H^n_i(f\psi_\varepsilon))^2\big) \le \overline{H}^n((f\psi_\varepsilon)^2)_t.$$

Observe that $(f\psi_\varepsilon)^2 \in \mathcal{E}'''_{2r} \cap C^{0,\nu}$, whereas $2r > 2$. Then $\overline{H}^n((f\psi_\varepsilon)^2) \stackrel{\text{u.c.p.}}{\longrightarrow} (f\psi_\varepsilon)^2 \star \nu$ by Lemma 3.1, hence

$$t > 0,\ \rho > 0 \quad \Rightarrow \quad \lim_{\varepsilon \to 0}\ \limsup_n\ \mathbb{P}(C^n(\varepsilon)_t > \rho) = 0. \tag{3.14}$$

By Lenglart inequality it is well known that (3.14) implies (3.13), and we are done.

4 Theorem 2.4

4.1 Technical consequences of (H).

The assumption (H) is "local", in the sense that it does not require any integrability assumptions (in ω) on the characteristics. However having "locally bounded" replaced by "bounded", for example, simplifies a lot of technical problems. This is why we introduce "global" and apparently much stronger conditions:

Hypothesis (SH): We have (H), and the processes (b_t), (c_t) and $(F_t(\phi_2))$ are bounded (by a – non-random – constant), and the jumps of X are also bounded by a constant. $\Box$

Next, we introduce a number of notation, for which we assume (H) and heavily use σ, as in (2.10). Recall $X' = X - X_0 - X^c$:

$$\left.\begin{array}{l} \chi'^n_i = \frac{1}{\sqrt{\Delta_n}} \int_{(i-1)\Delta_n}^{i\Delta_n} (\sigma_s - \sigma_{(i-1)\Delta_n})\, dW_s \\ \beta^n_i = \sigma_{(i-1)\delta_n} \Delta^n_i W/\sqrt{\Delta_n}, \qquad \chi^n_i = \chi'^n_i + \frac{1}{\sqrt{\Delta_n}}\, \Delta^n_i X' \\ \rho^n_i = \rho_{\sigma_{(i-1)\Delta_n}}. \end{array}\right\} \tag{4.1}$$

In particular, $\Delta_i^n X = \chi_i^n + \beta_i^n$. It is obvious that (SH) implies for all $q > 0$:

$$\left.\begin{array}{l}
\mathbb{E}_{i-1}^n(|\beta_i^n|^q) \leq K_q, \qquad \mathbb{E}_{i-1}^n(|\chi_i'^n|^q) \leq K_q, \qquad \mathbb{E}_{i-1}^n(|\Delta_i^n X^c|^q) \leq K_q \Delta_n^{q/2} \\
\mathbb{E}_{i-1}^n(|\Delta_i^n X'|^q) \leq \begin{cases} K_q \Delta_n^{1 \bigwedge (q/2)} & \text{in general} \\ K_q \Delta_n^q & \text{if } X \text{ is continuous} \end{cases} \\
\mathbb{E}_{i-1}^n(|\chi_i^n|^q) \leq \begin{cases} K_q \Delta_n^{-(1-q/2)^-} & \text{in general} \\ K_q & \text{if } X \text{ is continuous} \end{cases}
\end{array}\right\} \tag{4.2}$$

Lemma 4.1 *Under (SH) we have*

$$\Delta_n \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}(\phi_2(\chi_i^n)) \quad \stackrel{u.c.p.}{\longrightarrow} \quad 0. \tag{4.3}$$

Proof. For any $\varepsilon \in (0,1]$ we write $X' = N(\varepsilon) + M(\varepsilon) + B(\varepsilon)$, where

$$N(\varepsilon) = (x 1_{\{|x|>\varepsilon\}}) \star \mu, \quad M(\varepsilon) = (x 1_{\{|x|\leq\varepsilon\}}) \star (\mu - \nu), \quad B(\varepsilon) = B - (\kappa(x) 1_{\{|x|>\varepsilon\}}) \star \nu.$$

We also set

$$\gamma_i^n(y) = \frac{1}{\Delta_n}\, \mathbb{E}_{i-1}^n \left(\int_{(i-1)\Delta_n}^{i\Delta_n} dt \int_{\{|x|\leq y\}} \phi_2(x) F_t(dx) \right),$$

which is increasing in y with $\gamma_i^n(y) \leq K$ by (SH). Then

$$\mathbb{P}_{i-1}^n(\Delta_i^n N(\varepsilon) \neq 0) \leq K\varepsilon^{-2}\Delta_n, \quad \mathbb{E}_{i-1}^n((\Delta_i^n M(\varepsilon))^2) \leq \Delta_n \gamma_i^n(\varepsilon), \quad |\Delta_i^n B(\varepsilon)| \leq K\Delta_n \varepsilon^{-1}$$

(use Tchebycheff inequality for the first and last estimates). We also have

$$\mathbb{E}_{i-1}^n((\chi_i'^n)^2) \;=\; \gamma_i'^n \;:=\; \frac{1}{\Delta_n}\, \mathbb{E}_{i-1}^n \left(\int_{(i-1)\Delta_n}^{i\Delta_n} (\sigma_u - \sigma_{(i-1)\Delta_n})^2 \, du \right).$$

The following is obvious:

$$\phi_2(\chi_i^n) \leq 1_{\{\Delta_i^n N(\varepsilon)\neq 0\}} + 3|\chi_i'^n|^2 + 3\Delta_n^{-1}(|\Delta_i^n M(\varepsilon)|^2 + 3\Delta_n^{-1}|\Delta_i^n B(\varepsilon)|^2),$$

Then if we take $\varepsilon = \varepsilon_n = \Delta_n^{1/4}$ we deduce from the previous estimates that

$$\mathbb{E}_{i-1}^n\left(\phi_2(\chi_i^n)\right) \leq K\sqrt{\Delta_n} + K\gamma_i'^n + K\gamma_i^n(\varepsilon_n). \tag{4.4}$$

Now, observe that

$$\Delta_n \, \mathbb{E}\left(\sum_{i=1}^{[t/\Delta_n]} (\gamma_i^n(\varepsilon_n) + \gamma_i'^n) \right) \leq \mathbb{E}\left(\int_0^t du \left((\sigma_u - \sigma_{\Delta_n[u/\Delta_n]})^2 + \int_{\{|x|\leq\varepsilon_n\}} \phi_2(x) F_u(x) \right) \right).$$

(SH) implies that for each (ω, u) the middle parenthesis in the right side above goes to 0, while staying bounded by a constant, so by Lebesgue's theorem the left side goes to 0. Plugging this into (4.4) immediately gives (4.3). $\square$

Lemma 4.2 *Under (SH) we have for all $f \in \mathcal{E}_1''$ and all $\rho > 0$:*

$$\lim_{\varepsilon\to 0}\ \lim_{A\to\infty}\ \limsup_n\ \mathbb{P}\left(\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}_{i-1}^n\Big((f(\psi_\varepsilon - \psi_{A\sqrt{\Delta_n}}))^2(\Delta_i^n X)\Big) > \rho\right) = 0. \tag{4.5}$$

Proof. We have $|f(x)| \le K|x|$ for $|x| \le 1$, so as soon as $A\sqrt{\Delta_n} \le \varepsilon/2 \le 1/4$ we have (by singling out the two cases $|x| \le |y|$ and $|x| > |y|$):

$$\begin{aligned} |f(\psi_\varepsilon - \psi_{A\sqrt{\Delta_n}})|(x+y) &\le K|x|1_{\{A\sqrt{\Delta_n}/2 \le |x| \le 3\varepsilon\}} + K|y|1_{\{A\sqrt{\Delta_n}/2 \le |y| \le 3\varepsilon\}} \\ &\le K|x|(\psi_{3\varepsilon} - \psi_{A\sqrt{\Delta_n}/2})(x) + K|y|(\psi_{3\varepsilon} - \psi_{A\sqrt{\Delta_n}/2})(y). \end{aligned}$$

Hence it is enough to prove (4.5) for $f = h_1$, and separately for X^c and for X'. First, by (4.2) we have

$$\mathbb{E}_{i-1}^n(|\Delta_i^n X^c|^2(\psi_\varepsilon - \psi_{A\sqrt{\Delta_n}})^2(\Delta_i^n X^c)) \le \mathbb{E}_{i-1}^n(|\Delta_i^n X^c|^2 1_{\{|\Delta_i^n X^c| \ge A\sqrt{\Delta_n}\}}) \le \frac{K\Delta_n}{A},$$

and (4.5) for h_1 is then obvious for X^c. Second, we have

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}_{i-1}^n\Big(|\Delta_i^n X'|^2(\psi_\varepsilon - \psi_{A\sqrt{\Delta_n}})^2(\Delta_i^n X')\Big) \le \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}_{i-1}^n\Big((h_2\psi_\varepsilon)(\Delta_i^n X'')\Big).$$

Lemma 3.1 applied to $X = X'$ (note that $f\psi_\varepsilon$ is bounded) yields that the right side above converges u.c.p. to $(g\psi_\varepsilon) \star \nu_t$, and the later goes to 0 as $\varepsilon \to 0$: this shows (4.5) for X'. □

Lemma 4.3 *Under (H) we have $\Delta_n \sum_{i=1}^{[t/\Delta_n]} \rho_i^n(g) \overset{u.c.p.}{\longrightarrow} \int_0^t \rho_{\sigma_s}(g)ds$ if $g \in \mathcal{E}$ is continuous.*

(The continuity of g is much too strong for this, but the above result is enough for us).

Proof. The process σ is càdlàg, hence the function $s \mapsto \rho_s(g) = \mathbb{E}(g(\sigma_s U))$ is also càdlàg by Lebesgue's theorem. The result is then obvious by Riemann approximation of the integral. □

Lemma 4.4 *(i) Under (H) any even function g in $\mathcal{E}$ we have*

$$\mathbb{E}_{i-1}^n\,(\Delta_i^n N\ g(\beta_i^n)) \ = \ 0 \tag{4.6}$$

for $N = W$ and for all N in the set $\mathcal{N}$ of all continuous bounded martingales which are orthogonal to W.

(ii) Assume (SH), and let $g \in \mathcal{E}$ be continuous. If further $q > 0$, and $g(x)/|x|^{2/q} \to 0$ as $|x| \to \infty$ when X is not continuous, then

$$\Delta_n \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}_{i-1}^n\left(\Big|g(\Delta_i^n X/\sqrt{\Delta_n}) - g(\beta_i^n)\Big|^q\right) \ \overset{u.c.p.}{\longrightarrow}\ 0. \tag{4.7}$$

In particular, provided $g(x)/x^2 \to 0$ as $|x| \to \infty$ whenever X is not continuous, then

$$\Delta_n \overline{K}^n(g)_t \ \overset{u.c.p.}{\longrightarrow}\ \int_0^t \rho_{\sigma_s}(g)ds. \tag{4.8}$$

Proof. (i) When $N = W$, we have $\Delta_i^n N g(\beta_i^n) = h(\sigma_{(i-1)\Delta_n}, \Delta_i^n W)$ for a function $h(x,y)$ which is odd and with polynomial growth in y when g is even, so obviously (4.6) holds. When $N \in \mathcal{N}$, (4.6) is proved in Proposition 4.1 of [4].

(ii) Since $\mathbb{E}_{i-1}^n(g(\beta_i^n)) = \rho_i^n(g)$, (4.8) readily follows from (4.7) for $q = 1$ and from Lemma 4.3. As for (4.7), it amounts to the AN property of the array $(\zeta_i'^n)$ defined as follows:

$$\zeta_i'^n = \Delta_n \mathbb{E}_{i-1}^n(|\zeta_i^n|^q), \qquad \zeta_i^n = g(\Delta_i^n X/\sqrt{\Delta_n}) - g(\beta_i^n).$$

We first prove this result when $g(x)/|x|^{2/q} \to 0$ at infinity. Set $G_A(\varepsilon) = \sup(|g(x+y) - g(x)| : \; |x| \le A, \; |y| \le \varepsilon)$ and $H_A = \sup_{|x|>A} |g(x)|/|x|^{2/q}$ and $L_A = \sup_{|x| \le A} |g(x)|$. We have $G_A(\varepsilon) \to 0$ as $\varepsilon \to 0$ for all A, and $H_A \to 0$ as $A \to \infty$, and $L_A < \infty$ for all A, and also $|g(x+y)| \le L_B + K_q H_B(|x|^{2/q} + |y|^{2/q})$ for all $B > 0$. Then, since $\Delta_i^n X/\sqrt{\Delta_n} = \beta_i^n + \chi_i^n$, and with the notation $W_i^n = \Delta_i^n W/\sqrt{\Delta_n}$, we obtain for $\varepsilon \in (0,1]$, $A, B > 0$, and if $|\sigma| \le \Gamma$:

$$|\zeta_i^n| \le K \left(G_A(\varepsilon) + H_B \Big(|\chi_i^n|^{2/q} + |\Gamma W_i^n|^{2/q} \Big) + L_B \left(1_{\{|W_i^n| > A/\Gamma\}} + \varepsilon^{-2/q} \; \phi_{2/q}(\chi_i^n) \right) \right),$$

and thus by using (4.2),

$$\zeta_i'^n \le K \Delta_n \left(G_A(\varepsilon)^q + H_B^q + L_B^q \mathbb{P}(|U| > A/\Gamma) + L_B^q \varepsilon^{-2} \mathbb{E}_{i-1}^n(\phi_2(\chi_i^n)) \right).$$

Therefore if we use (4.3) we get

$$\begin{aligned} \sum_{i=1}^{[t/\Delta_n]} \zeta_i'^n \; &\le \; Kt \left(G_A(\varepsilon)^q + H_B^q + L_B^q \mathbb{P}(|U| > A/\Gamma) \right) + K L_B^q \Delta_n \varepsilon^{-2} \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}_{i-1}^n(\phi_2(\chi_i^n)) \\ &\stackrel{\text{u.c.p.}}{\longrightarrow} \; Kt \Big(G_A(\varepsilon)^q + H_B^q + L_B^q \mathbb{P}(|U| > A/\Gamma) \Big). \end{aligned}$$

Then we take B such that H_B is small, then A such that $L_B \mathbb{P}(|U| > A/\Gamma)$ is small, than ε such that $G_A(\varepsilon)$ is small, and we deduce that the array $(\zeta_i'^n)$ is AN.

Finally when X is continuous and g is of polynomial growth, we have $H_A = \infty$, but since now $\chi_i^n = \chi_i'^n$ we can use the estimate $|g(x)| \le K(1 + |x|^p)$ for some $p > 0$ to get

$$|\zeta_i^n| \le K \left(G_A(\varepsilon) + \Big(1 + |\chi_i'^n|^p + |W_i^n|^p \Big) \left(1_{\{|W_i^n| > A/\Gamma\}} + \varepsilon^{-1} \; |\chi_i'^n| \right) \right),$$

hence by Hölder and (4.2) we deduce

$$\zeta_i'^n \le K \Delta_n \left(G_A(\varepsilon)^q + (\mathbb{P}(|U| > A/\Gamma))^{1/2} + \mathbb{E}(|U|^p 1_{\{|U| > A/\Gamma\}}) + \frac{1}{\varepsilon} \; \left(\mathbb{E}_{i-1}^n(|\chi_i'^n|^2) \right)^{1/2} \right).$$

Then we may conclude as above, using Lemma 7.8 of [4] instead of (4.4). □

4.2 Proof of Theorem 2.4.

This theorem is a consequence of the following two lemmas: the first one proves the result under the stronger assumptions (SH), the second one is a standard localization procedure giving the results under (H).

Lemma 4.5 *Theorem 2.4 holds under the assumption (SH).*

Proof. (i) If $g \in \mathcal{E}$ is continuous, the process

$$\overline{V}^n(g)_t = \Delta_n \sum_{i=1}^{[t/\Delta_n]} \Big(g(\beta_i^n) - \rho_i^n(g) \Big).$$

is a square-integrable martingale w.r.t. the filtration $(\mathcal{F}'^n_t = \mathcal{F}_{\Delta_n[t/\Delta_n]})_{t \geq 0}$, and its predictable bracket is $\Delta_n^2 \sum_{i=1}^{[t/\Delta_n]} \Big(\rho_i^n(g^2) - \rho_i^n(g)^2 \Big) \leq K t \Delta_n$. Hence $\overline{V}^n(g) \stackrel{\text{u.c.p.}}{\longrightarrow} 0$ and we deduce from Lemma 4.3 that

$$\Delta_n \sum_{i=1}^{[t/\Delta_n]} g(\beta_i^n) \stackrel{\text{u.c.p.}}{\longrightarrow} \int_0^t \rho_u(g) du. \tag{4.9}$$

If further $g(x)/x^2 \to 0$ as $|x| \to \infty$, or if X is continuous, we can apply (4.7) with $q = 1$ to deduce via Lenglart's inequality that

$$\Delta_n \sum_{i=1}^{[t/\Delta_n]} \Big(g(\Delta_i^n X/\sqrt{\Delta_n}) - g(\beta_i^n) \Big) \stackrel{\text{u.c.p.}}{\longrightarrow} 0.$$

Combining this with (4.9) gives (i).

(ii) We have $\Delta_n^{-r/2} V^n(h_r) = V'^n(h_r)$. If $f \in \mathcal{E}_r$, for all $\varepsilon > 0$ we also have $|f - h_r| \leq \varepsilon h_{qr} + K_\varepsilon (1 - \psi_\varepsilon) h_p$ for some constants $p > 2$ and $K_\varepsilon > 0$, hence

$$|\Delta_n^{1-r/2} V^n(f) - \Delta_n V'^n(h_r)| \leq \varepsilon \Delta_n V'^n(h_r) + \Delta_n^{1-r/2} V^n((1 - \psi_\varepsilon) h_p). \tag{4.10}$$

On the one hand $(1 - \psi_\varepsilon) h_p \in \mathcal{E}_2''' \cap C^0$, so Theorem 2.2 and $r < 2$ yield $\Delta_n^{1-r/2} V^n((1 - \psi_\varepsilon) h_p) \stackrel{\text{u.c.p.}}{\longrightarrow} 0$. On the other hand $\Delta_n V'(h_r) \stackrel{\text{u.c.p.}}{\longrightarrow} m_r \int_0^t c_u^{r/2}\, du$ by (i). Since $\varepsilon > 0$ is arbitrarily small, we deduce (ii) from (4.10).

(iii) For any $0 < \varepsilon < A < \infty$ we have $2A\sqrt{\Delta_n} \leq \alpha \Delta_n^{\varpi} \leq \varepsilon$ for all n large enough, and if this holds we have

$$\Delta_n V'^n(h_2 \psi_A) \leq V''^n(\varpi, \alpha) \leq V^n(h_2 \psi_\varepsilon). \tag{4.11}$$

Since $h_2 \psi_A$ is bounded continuous, (i) implies that the left side above converges u.c.p. to $V'(A)_t = \int_0^t \rho_{\sigma_u}(h_2 \psi_A)\, du$, which in turn increases (u.c.p. again) to C_t as $A \to \infty$. The right side of (4.11) has jumps smaller than $4\varepsilon^2$, and since $h_2 \psi_\varepsilon \in \mathcal{E}_2 \cap C^0$ it converges in probability for the Skrokhod topology to $C + (h_2 \psi_\varepsilon) \star \mu$ by Theorem 2.2-(b), whereas $C + (h_2 \psi_\varepsilon) \star \mu$ decreases u.c.p. to C as $\varepsilon \to 0$. Then (iii) is obvious. $\Box$

Lemma 4.6 *If Theorem 2.4 holds under the assumption (SH), it also holds under the assumption (H).*

Proof. (H) implies the existence of a sequence of stopping times T_p increasing to ∞ and such that the three processes (b_t), (c_t) and $(F_t(\phi_2))$ are bounded by a constant K_p for all $t \leq T_p$, and also such that $|\Delta X_s| \leq p$ for all $s < T_p$ (note that we usually cannot find T_p as above, such that $|\Delta X_{T_p}| \leq p$). Then the process

$$X(p)_t = X_0 + B_{t \bigwedge T_p} + X^c_{t \bigwedge T_p} + \kappa \star (\mu - \nu)_{t \bigwedge T_p} + (\kappa'(x) 1_{\{|x| \leq p\}}) \star \mu_{t \bigwedge T_p}$$

(compare with (2.4)) satisfies (2.9) with $b(p)_t = b_t 1_{\{t \leq T_p\}}$ and $c(p)_t = c_t 1_{\{t \leq T_p\}}$ and $F(p)_t(dx) = 1_{\{|x| \leq p\}} \cdot F_t(dx) 1_{\{t \leq T_p\}}$, and also $|\Delta X(p)| \leq p$ by construction: hence $X(p)$ satisfies (SH).

By hypothesis, for each p the processes $\Delta_n V'^n(X(p); g)$ in (i) converge u.c.p. to $\int_0^t \rho_{\sigma(p)_u}(g) du = \int_0^{t \bigwedge T_p} \rho_{\sigma_u}(g) du$. Since we have $V'^n(X(p; g)_t = V'^n(X; g)_t$ for $t < T_p$, whereas $T_p \uparrow \infty$, we readily deduce the result for (i). For (ii) and (iii) it is proved in the same way. $\square$

5 Proofs for the CLTs

5.1 Technical consequences of (K), (L-s) and (H').

Exactly as for Assumption (H) which was strengthened into (SH) for technical reasons, we need to strengthen (K), (L-s) and (H') as follows :

Hypothesis (SK): We have (K) and (SH), and the functions $\gamma_k = \gamma$ do not depend on k and are bounded. $\square$

Hypothesis (SL-s): We have (L-s) and the processes (b_t), (c_t), $(\widetilde{b}_t)$, $(\widetilde{\sigma}_t)$, $(\widetilde{\sigma}'_t)$ are bounded, and the functions $\gamma_k = \gamma$ and $\widetilde{\gamma}_k = \widetilde{\gamma}$ do not depend on k and are bounded. $\square$

Hypothesis (SH'): We have (H) and the process (c_t) is bounded away from 0. $\square$

Now we proceed to derive some consequences of these assumptions, except that the first result, used for Theorem 2.8, needs (SH) only.

Lemma 5.1 *Assume (SH). If $f \in \mathcal{E}'_1$ we have for all $\rho > 0$:*

$$\lim_{\varepsilon \to 0} \limsup_n \mathbb{P}\left(\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}^n_{i-1} \left(\left| f(\Delta^n_i X) \psi_\varepsilon(\Delta^n_i X) - |\sqrt{\Delta_n}\ \beta^n_i| \right|^2 \right) > \rho \right) = 0. \tag{5.1}$$

Proof. Suppose first that $f = h_1$. Observe that for any $A > 0$,

$$\left| f(\Delta^n_i X) \psi_\varepsilon(\Delta^n_i X) - |\sqrt{\Delta_n}\ \beta^n_i| \right| \leq \sum_{j)1}^{3} \zeta^n_i(A, \varepsilon, j),$$

where, with the notation $g_A = f\psi_A$,

$$\zeta_i^n(A,\varepsilon,1) = \left| f(\Delta_i^n X)(\psi_\varepsilon(\Delta_i^n X) - \psi_{A\sqrt{\Delta_n}})(\Delta_i^n X) \right|,$$

$$\zeta_i^n(A,\varepsilon,2) = \sqrt{\Delta_n}\ \left| g_A(\Delta_i^n X/\sqrt{\Delta_n}) - g_A(\beta_i^n) \right|,$$

$$\zeta_i^n(A,\varepsilon,3) = \zeta_i^n(A,3) = \sqrt{\Delta_n}\ |\beta_i^n|\ (1-\psi_A(\beta_i^n)).$$

Then it is enough to show that

$$\lim_{\varepsilon\to 0}\ \lim_{A\to\infty}\ \limsup_n\ \mathbb{P}\left(\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}_{i-1}^n(\zeta_i^n(A,\varepsilon,j)^2) > \rho\right) = 0 \tag{5.2}$$

for all $\rho > 0$ and $j = 1,2,3$. Now, (5.2) for $j = 1$ is exactly (4.5), and (5.2) for $j = 2$ follows from (4.7) applied to g_A (which is bounded continuous) and $q = 2$, and (5.2) for $j = 3$ immediately follows from (4.2).

Finally when $f \in \mathcal{E}_1'$, in order to get the result it suffices to prove that the array

$$\zeta_i^n(\varepsilon) = \mathbb{E}_{i-1}^n\left(\left(\ \left|(f-h_1)(\Delta_i^n X)\right| \psi_\varepsilon(\Delta_i^n X)\right)^2\right)$$

is AN, for each $\varepsilon > 0$. We have $|(f-h_1)\psi_\varepsilon| \leq \eta_\varepsilon \phi_1$ with $\eta_\varepsilon \to 0$ as $\varepsilon \to 0$, hence

$$\mathbb{E}\left(\sum_{i=1}^{[t/\Delta_n]} \zeta_i^n\right) \leq \eta_\varepsilon^2\ \overline{H}_t^n(\phi_2),$$

and the result follows from Lemma 3.1. $\square$

Note that under (K), resp. (SL-2), Equations (2.11) and (2.12) take the form

$$X_t = X_0 + \int_0^t b_s' ds + \int_0^t \sigma_s dW_s + \delta \star (\underline{\mu} - \underline{\nu})_t, \tag{5.3}$$

$$\sigma_t = \sigma_0 + \int_0^t \widetilde{b}_s' ds + \int_0^t \widetilde{\sigma}_s dW_s + \int_0^t \widetilde{\sigma}_s' dW_s' + \widetilde{\delta} \star (\underline{\mu} - \underline{\nu})_t, \tag{5.4}$$

where $b_t' = b_t + \int \kappa'(\delta(t,x))\ dx$ and $\widetilde{b}_t' = \widetilde{b}_t + \int \kappa'(\widetilde{\delta}(t,x))\ dx$ are bounded.

The key result is that, under appropriate assumptions, the convergence in (4.8) holds with a rate $1/\sqrt{\Delta_n}$. This has been shown in [4] when X is continuous, i.e. $\delta = 0$, but in general we need some estimate on the increments of the process $\delta \star (\underline{\mu} - \underline{\nu})$. However we start with a result proved in Section 8-2 of [4], for which the process $\widetilde{\delta}$ plays no role:

Lemma 5.2 *Assume (SL-2) and let either g be differentiable with $g' \in \mathcal{E}$, or $g \in \mathcal{E}$ and (SH') hold. Then* $\frac{1}{\sqrt{\Delta_n}}\left(\sum_{i=1}^{[t/\Delta_n]} \Delta_n \rho_i^n(g) - \int_0^t \rho_{\sigma_s}(f) ds\right) \xrightarrow{u.c.p.} 0.$

Now we fix a sequence ε_n in $(0,1)$, going to 0 and to be chosen later, and we put $E_n = \{x : \gamma(x) > \varepsilon_n\}$. Recall that $t \mapsto \delta(\omega,t,x)$ is left continuous with right limits, and we denote by $\delta_+(\omega,t,x)$ the right limit at time t. Then we set

$$\left.\begin{array}{l}
\zeta_i^n(1) = \frac{1}{\sqrt{\Delta_n}} \int_{(i-1)\Delta_n}^{i\Delta_n} \int_{E_n^c} \delta(v,x)(\underline{\mu} - \underline{\nu})(dv,dx) \\
\zeta_i^n(2) = -\frac{1}{\sqrt{\Delta_n}} \int_{(i-1)\Delta_n}^{i\Delta_n} \int_{E_n} \delta_+((i-1)\Delta_n, x)\underline{\nu}(dv,dx) \\
\zeta_i^n(3) = -\frac{1}{\sqrt{\Delta_n}} \int_{(i-1)\Delta_n}^{i\Delta_n} \int_{E_n} (\delta(v,x) - \delta_+((i-1)\Delta_n, x))\underline{\nu}(dv,dx) \\
\zeta_i^n(4) = \frac{1}{\sqrt{\Delta_n}} \int_{(i-1)\Delta_n}^{i\Delta_n} \int_{E_n} \delta(v,x)\underline{\mu}(dv,dx),
\end{array}\right\} \tag{5.5}$$

We denote by $\mathcal{F}'^n_i$ the σ-field generated by $\mathcal{F}_{(i-1)\Delta_n}$ and the variables $(W_u : 0 \le u \le i\Delta_n)$.

Lemma 5.3 *Assume (SL-s). The function $\gamma_s(y) = \int_{\{x:\gamma(x)\le y\}} \gamma(x)^s dx$ is bounded increasing and goes to 0 as $y \to 0$, and we have for $r \in (0,1]$ and $\alpha > 0$:*

$$\mathbb{E}\left(\zeta_i^n(1)^2 \mid \mathcal{F}'^n_i\right) \le K\varepsilon_n^{2-s}\gamma_s(\varepsilon_n), \tag{5.6}$$

$$|\zeta_i^n(2)| + |\zeta_i^n(3)| \le K\sqrt{\Delta_n}\, \varepsilon_n^{-(s-1)^+}, \tag{5.7}$$

$$\mathbb{E}\left(|\zeta_i^n(4)|^r \mid \mathcal{F}'^n_i\right) \le K\Delta_n^{1-r/2}\varepsilon_n^{-(s-r)^+}, \tag{5.8}$$

$$\mathbb{E}\left(|\zeta_i^n(4)| \bigwedge \alpha \mid \mathcal{F}'^n_i\right) \le K\alpha\Delta_n\varepsilon_n^{-s}. \tag{5.9}$$

Proof. Obviously $|\zeta_i^n(2)| + |\zeta_i^n(3)| \le 3\sqrt{\Delta_n}\int_{E_n}\gamma(x)dx$, hence Tchebycheff's inequality yields (5.7). Next, conditionally on $\mathcal{F}'^n_i$, the measure $\underline{\mu}$ restricted to $((i-1)\Delta_n, \infty) \times \mathbb{R}$ is still a Poisson measure with intensity measure $\underline{\nu}$, because $\underline{\mu}$ and W are independent. Hence

$$\mathbb{E}(\zeta_i^n(1)^2 \mid \mathcal{F}'^n_i) = \frac{1}{\Delta_n}\mathbb{E}\left(\int_{(i-1)\Delta_n}^{i\Delta_n} du \int_{E_n^c} \delta(u,x)^2 dx \Big| \mathcal{F}'^n_i\right) \le \int_{E_n^c} \gamma(x)^2 dx,$$

and (5.6) follows.

Finally $|\zeta_i^n(4)| \le Z_i^n := \frac{1}{\sqrt{\Delta_n}} \int_{(i-1)\Delta_n}^{i\Delta_n} \int_{E_n} \gamma(x)\underline{\mu}(dv,dx)$, and Z_i^n is independent of $\mathcal{F}'^n_i$ and is a compound Poisson variable. More specifically, if $\eta_n = \int 1_{E_n}(x)dx$, then Z_i^n is the sum of N i.i.d. variables Y_j with $\mathbb{E}(f(Y_j)) = \frac{1}{\eta_n}\int_{E_n} f(\gamma(x)/\sqrt{\Delta_n})dx$ and N is independent of the Y_j's and Poisson with parameter $\eta_n\Delta_n$. We deduce first that

$$\mathbb{E}(|Z_i^n|^r) \le \mathbb{E}\Big(\sum_{j=1}^N |Y_j|^r\Big) = \mathbb{E}(N)\ \mathbb{E}(|Y_1|^r) = \Delta_n^{1-r/2}\int_{E_n} \gamma(x)^r dx,$$

and second that

$$\mathbb{E}(|Z_i^n| \bigwedge \alpha) \le \alpha\mathbb{P}(N \ge 1) \le \Delta_n\eta_n.$$

Since $\int \gamma(x)^s dx < \infty$, we deduce (5.8) and (5.9) from Tchebycheff's inequality again. $\square$

Lemma 5.4 *Under (SL-s) we have*

$$\sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} \sqrt{\mathbb{E}_{i-1}^n(|\zeta_i^n(3)|^2)} \;=\; \mathrm{o}_{Pu}(\varepsilon_n^{-(s-1)^+}). \tag{5.10}$$

Proof. By a repeated application of Cauchy-Schwarz, the expected value of the left side of (5.10), say $a_n(t)$, satisfies

$$\begin{aligned}
a_n(t)^2 \;&\leq\; t\mathbb{E}\left(\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}_{i-1}^n(|\zeta_i^n(3)|^2)\right) \\
&\leq t\mathbb{E}\left(\sum_{i=1}^{[t/\Delta_n]} \frac{1}{\Delta_n}\left(\int_{(i-1)\Delta_n}^{i\Delta_n} dv \int_{E_n} |\delta(v,x) - \delta_+((i-1)\Delta_n, x))|x\right)^2\right) \\
&\leq t\mathbb{E}\left(\int_0^t dv \int_{E_n} |\delta(v,x) - \delta_+(\Delta_n[v/\Delta_n], x)|^s dx \int_{E_n} |\delta(v,x) - \delta_+(\Delta_n[v/\Delta_n], x)|^{2-s} dx\right) \\
&\leq t\eta_n(s)\mathbb{E}\left(\int_0^t dv \int_{E_n} |\delta(v,x) - \delta_+(\Delta_n[v/\Delta_n], x)|^s dx\right),
\end{aligned}$$

where $\eta_n(s) = \int_{E_n} (2\gamma(x))^{2-s} dx$. Now, on the one hand the expectation in the last term above goes to 0 because of the properties of δ and of (SL-s), by an application of Lebesgue's theorem. On the other hand we have $\eta_n(s) \leq K$ is $s \leq 1$, and when $s > 1$ we have $\eta_n(s) \leq K\varepsilon_n^{2-2s}$ by Tchebycheff's inequality, and the result follows. $\square$

We are now ready to improve on (4.8) by giving a rate, at least in some special situations. That is, we give estimates on the processes

$$U^n(g)_t \;=\; \Delta_n \overline{K}^n(g)_t - \int_0^t \rho_{\sigma_u}(g)\, du \tag{5.11}$$

in three different situations:

- Case (i): g is C_b^2,
- Case (ii): $g_n = h_2\, \psi_{\alpha\Delta_n^{\varpi-1/2}}$ for some $\alpha > 0$ and some $\varpi \in (0, 1/2)$,
- Case (iii): $g = h_r$ for some $r \in (0,1)$,

and also when g is C^1 with $g' \in \mathcal{E}$, when X is continuous. We also need to introduce the following functions η_s on $[0,1]$, where $s \in [0,2]$:

$$\eta_s(r) = \frac{(2-s)(1+r)(2-r)}{4 + 2s(1-r)}. \tag{5.12}$$

Lemma 5.5 *Assume (SL-s). Then*

$$\frac{1}{\sqrt{\Delta_n}}\, U^n(g) \;\stackrel{u.c.p.}{\longrightarrow}\; 0, \qquad or \qquad \frac{1}{\sqrt{\Delta_n}}\, U^n(g_n) \;\stackrel{u.c.p.}{\longrightarrow}\; 0 \tag{5.13}$$

in the following cases:

(a) X is continuous and either g is C^1 with $g' \in \mathcal{E}$ and g is even, or $g = h_r$ for $r \in (0,1]$ if further (SH') holds;

(b) in case (i) if $s \leq 1$;

(c) in case (ii) if $0 \leq s \leq \frac{4\varpi-1}{2\varpi}$ (hence $\varpi \geq \frac{1}{4}$ and $s < 1$);

(d) in case (iii) if further (SH') holds, and provided either $s \leq \frac{2}{3}$ and $0 < r < 1$, or $\frac{2}{3} < s < 1$ and $\frac{1-\sqrt{3s^2-8s+5}}{2-s} < r < 1$,

Otherwise, we have for all $\varepsilon > 0$:

$$U^n(g_n)_t = \begin{cases} \mathrm{o}_{Pu}(\Delta_n^{1-s/2}) & \text{in case (i) if } s > 1 \\ \mathrm{o}_{Pu}(\Delta_n^{(2-s)\varpi}) & \text{in case (ii) if } s > \frac{4\varpi-1}{2\varpi} \\ \mathrm{O}_{Pu}(\sqrt{\Delta_n}) & \text{if } g_n = h_1, \text{ (SH') holds and } s \leq 1 \\ \mathrm{o}_{Pu}(\Delta_n^{\eta_s(r)-\varepsilon}) & \text{if } g_n = h_r \text{ and (SH') holds and} \\ & \text{and either } 0 < r \leq 1 < s, \text{ or } \frac{2}{3} < s < 1 \text{ and} \\ & 0 < r \leq \frac{1-\sqrt{3s^2-8s+5}}{2-s} \text{ and } r < 1. \end{cases} \tag{5.14}$$

In the last case of (5.14) we have $0 < \eta_s(r) < 1/2$. Comparing with (5.14) of [4], the case (b) above and the first estimate in (5.14) are just as good, except for the regularity conditions on g; but we suspect that the last two estimates in (5.14) are not optimal: when X is a Lévy process we have $U^n(h_r)_t = \mathrm{o}_{Pu}(\sqrt{\Delta_n})$ as soon as $s \leq 1$ and $r < 1$. The same comment will also apply to Lemma 5.6 below. Recall that when X is continuous the assumptions (SL-s) for $s \in [0,2]$ are all equivalent.

Proof. Throughout, we assume (SL-s). Since (a) is in [4], we only consider (b), (c), (d) and (5.14), with $g_n = g$ in cases (i) and (ii), and we set $U_t^n = U^n(g_n)_t$ and $\alpha_n = \alpha\Delta_n^{\varpi-1/2}$. The proof goes through several steps.

a) First, we state some obvious properties of our functions g_n :

$$\left.\begin{array}{l} |g(x+y) - g(x)| \leq K(|y| \bigwedge 1) \\ |g(x+y) - g(x) - g'(x)y| \leq K(|y| \bigwedge y^2) \end{array}\right\} \quad \text{in case (i)}, \tag{5.15}$$

$$\left.\begin{array}{l} |g_n(x+y) - g_n(x)| \leq K\alpha_n(|y| \bigwedge \alpha_n) \\ |g_n(x+y) - g_n(x) - g_n'(x)y| \leq Ky^2 \end{array}\right\} \quad \text{in case (ii)}, \tag{5.16}$$

$$\left.\begin{array}{l} |g_n(x+y) - g_n(x)| \leq K|y|^r \\ x \neq 0 \quad \Rightarrow \quad |g_n'(x)| \leq K|x|^{r-1} \\ 0 < |y| \leq \frac{|x|}{2} \quad \Rightarrow \quad |g_n'(x+y) - g_n'(x)| \leq K|x|^{r-2}|y| \end{array}\right\} \quad \text{in case (ii)}, \tag{5.17}$$

b) Recall (4.1) and set $\beta_i'^n = \beta_i^n + \chi_i^n - \zeta_i^n(4)$. Using the previous estimates, we readily deduce from Lemma 5.3 that (recall the notation (5.5)):

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}_{i-1}^n\left(\left|g_n(\beta_i^n + \chi_i^n) - g_n(\beta_i'^n)\right|\right) = \begin{cases} \mathrm{O}_{Pu}(\varepsilon_n^{-s}) & \text{in case (i)} \\ \mathrm{O}_{Pu}(\alpha_n^2\varepsilon_n^{-s}) & \text{in case (ii)} \\ \mathrm{O}_{Pu}(\Delta_n^{-r/2}\varepsilon_n^{-(s-r)^+}) & \text{in case (iii)} \end{cases} \tag{5.18}$$

c) Now we state some results of [4], see in particular Lemma 7.7 : put $\widehat{\beta}_i^n = 1 + |\beta_i^n|^{-1}$ and $Z_i^n = 1 + |\beta_i^n|^q$ for some $q \geq 0$. Then under (SH'), for any $\theta \in (1,2)$ and $l \in (0,1)$ the variables $\chi_i^n - \sum_{j=1}^4 \zeta_i^n(j)$ (in which the jumps of X play no role) are of the form $\widehat{\xi}_i^n + \widetilde{\xi}_i^n$, a decomposition which depends on θ and l and which satisfies

$$\left.\begin{array}{l} \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}_{i-1}^n(Z_i^n \, |\widehat{\beta}_i^n|^l \, |\widetilde{\xi}_i^n|^\theta) = \mathrm{O}_{Pu}(\Delta_n^{\theta/2-1}) \\ \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}_{i-1}^n(Z_i^n \, |\widehat{\beta}_i^n|^l \, |\widehat{\xi}_i^n|) = \mathrm{o}_{Pu}(\Delta_n^{-1/2}), \end{array}\right\} \tag{5.19}$$

and also for all odd functions k in $\mathcal{E}$:

$$\mathbb{E}_{i-1}^n(k(\beta_i^n)\widetilde{\xi}_i^n) = 0. \tag{5.20}$$

Furthermore a look at the proof of the afore-mentioned lemma shows that when $l = 0$ these hold also without (SH') and for $\theta = 2$, and that (again without (SH'))

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}_{i-1}^n(|\widehat{\xi}_i^n|^2) = \mathrm{O}_{Pu}(1). \tag{5.21}$$

In cases (i) we set $\theta = 2$ and $l = 0$ and $Z_i^n = 1$. In case (ii) we set $\theta = 2$ and $l = 0$ and $Z_i^n = 1 + |\beta_i^n|$. In case (iii) we fix $\theta \in (1, r+1)$ (to be chosen later), and set $l = \theta - r$ which is in $[0,1)$, and $Z_i^n = 1 + |\beta_i^n|^{\theta-1}$.

Now we set $\widetilde{\xi}_i'^n = \widetilde{\xi}_i^n + \sum_{j=1}^3 \zeta_i^n(j)$, so that $\beta_i'^n = \beta_i^n + \widehat{\xi}_i^n + \widetilde{\xi}_i'^n$. Since $\mathbb{E}_{i-1}^n(Z_i^n \, |\widehat{\beta}_i^n|^l) \leq K$ when l is as above, and since Z_i^n and $\widehat{\beta}_i^n$ are $\mathcal{F}_i'^n$-measurable, we deduce from (5.6), (5.7) and (5.19) that (with $l = 0$ in cases (i,ii), and l as above, under (SH'), in case (iii)) :

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}_{i-1}^n(Z_i^n \, |\widehat{\beta}_i^n|^l \, |\widetilde{\xi}_i'^n|^\theta) = \mathrm{O}_{Pu}\left(\Delta_n^{\theta/2-1}\varepsilon_n^{-\theta(s-1)^+} + \Delta_n^{-1}\varepsilon_n^{\theta(2-s)/2}\gamma_s(\varepsilon_n)^{\theta/2}\right). \tag{5.22}$$

If $k \in \mathcal{E}$ is odd and since $\zeta_i^n(2)$ is $\mathcal{F}_{(i-1)\Delta_n}$-measurable, we clearly have $\mathbb{E}_{i-1}^n(g'(\beta_i^n)\zeta_i^n(j)) = 0$ for $j = 2$, whereas this also holds for $j = 1$ because, as in the proof of Lemma 5.3, we have $\mathbb{E}(\zeta_i^n(1) \mid \mathcal{F}_i'^n) = 0$. Furthermore we have $|\mathbb{E}_{i-1}^n(k(\beta_i^n)\zeta_i^n(3))| \leq K\sqrt{\mathbb{E}_{i-1}^n(|\zeta_i^n(3)|^2)}$ by Cauchy-Schwarz, and because $\mathbb{E}_{i-1}^n(k(\beta_i^n)^2) \leq K$ by (4.2). Combining these facts with (5.20) and (5.10), we obtain

$$\sum_{i=1}^{[t/\Delta_n]} \left|\mathbb{E}_{i-1}^n(k(\beta_i^n)\widetilde{\xi}_i'^n)\right| \;=\; \mathrm{o}_{Pu}(\Delta_n^{-1/2}\varepsilon_n^{-(s-1)^2}). \tag{5.23}$$

d) Now, to evaluate $g_n(\beta_i'^n) - g_n(\beta_i^n)$, we partly reproduce a proof in [4], with some relevant changes. We first treat case (iii), which is the most difficult. Set $A_i^n = \{|\widetilde{\xi}_i'^n + \widehat{\xi}_i^n| > |\beta_i^n|/2\}$. By a Taylor expansion, we have $g(\beta_i'^n) - g(\beta_i^n) = g'(\beta_i^n)\widetilde{\xi}_i'^n + \sum_{j=1}^4 \delta_i^n(j)$, where

$$\delta_i^n(1) = (g(\beta_i'^n) - g(\beta_i^n))1_{A_i^n}, \qquad \delta_i^n(2) = -g'(\beta_i^n)(\beta_i'^n - \beta_i^n)1_{A_i^n},$$

$$\delta_i^n(3) = g'(\beta_i^n)\widehat{\xi}_i^n, , \qquad \delta_i^n(4) = (g'(\beta_i''^n) - g'(\beta_i^n))(\beta_i'^n - \beta_i^n)1_{(A_i^n)^c},$$

where $\beta_i''^n$ is a random variable which is between β_i^n and $\beta_i'^n$.

If we apply (5.17) and single out the two cases $|\widetilde{\xi}_i'^n| \geq |\widehat{\xi}_i^n|$ and $|\widetilde{\xi}_i'^n| < |\widehat{\xi}_i^n|$, upon observing that in the first case for instance we have $|\widetilde{\xi}_i'^n| > |\beta_i^n|/4$ if we are on A_i^n, we get

$$\begin{aligned} |\delta_i^n(1)| &\leq K|\widetilde{\xi}_i'^n + \widehat{\xi}_i^n|^r)1_{A_i^n} \\ &\leq K|\widehat{\xi}_i^n|\ |\beta_i^n|^{r-1} + K|\widetilde{\xi}_i'^n|^\theta\ |\beta_i^n|^{r-\theta} \ \leq\ KZ_i^n\ |\widehat{\beta}_i^n|^l\left(|\widehat{\xi}_i^n| + |\widetilde{\xi}_i'^n|^\theta\right) \end{aligned}$$

(recall $\theta \geq r$). In a similar way,

$$|\delta_i^n(2)| + |\delta_i^n(3)| \ \leq\ K|\widehat{\xi}_i^n|\ |\beta_i^n|^{r-1} + K|\widetilde{\xi}_i^n|^\theta\ |\beta_i^n|^{r-\theta} \ \leq\ KZ_i^n\ |\widehat{\beta}_i^n|^l\left(|\widehat{\xi}_i^n| + |\widetilde{\xi}_i'^n|^\theta\right).$$

Finally, by singling out the cases $|\widetilde{\xi}_i'^n| \geq |\widehat{\xi}_i^n|$ and $|\widetilde{\xi}_i'^n| < |\widehat{\xi}_i^n|$ once more,

$$\begin{aligned} |\delta_i^n(4)| &\leq K|\beta_i^n|^{r-2}\ (\widetilde{\xi}_i'^n + \widehat{\xi}_i^n)^2 1_{\{|\widetilde{\xi}_i'^n + \widehat{\xi}_i^n| \leq |\beta_i^n|/2\}} \\ &\leq K|\beta_i^n|^{r-2}\ \left(|\widehat{\xi}_i^n|\ |\beta_i^n| + |\widetilde{\xi}_i'^n|^\theta\ |\beta_i^n|^{2-\theta}\right) \ \leq\ KZ_i^n\ |\widehat{\beta}_i^n|^l\left(|\widehat{\xi}_i^n| + |\widetilde{\xi}_i'^n|^\theta\right). \end{aligned}$$

Put these three estimates together to get

$$\left|g(\beta_i'^n) - g(\beta_i^n) - g'(\beta_i^n)\widetilde{\xi}_i'^n\right| \leq KZ_i^n\ |\widehat{\beta}_i^n|^l\ \left(|\widehat{\xi}_i^n| + |\widetilde{\xi}_i'^n|^\theta\ \right). \tag{5.24}$$

In case (i) things are easier. Indeed, (5.15) implies:

$$\begin{aligned} \left|g(\beta_i'^n) - g(\beta_i^n) - g'(\beta_i^n)\widetilde{\xi}_i'^n\right| &\leq K(|\widetilde{\xi}_i'^n + \widehat{\xi}_i^n|^2 \bigwedge |\widetilde{\xi}_i'^n + \widehat{\xi}_i^n|) + K|\widehat{\xi}_i^n| \\ &\leq K(|\widehat{\xi}_i^n| + |\widetilde{\xi}_i'^n|^2). \end{aligned} \tag{5.25}$$

In case (ii), we use the fact that $|g_n'(x)| \leq K|x|)$ and (5.16) to get :

$$\begin{aligned} \left|g_n(\beta_i'^n) - g_n(\beta_i^n) - g_n'(\beta_i^n)\widetilde{\xi}_i'^n\right| &\leq K|\widetilde{\xi}_i'^n + \widehat{\xi}_i^n|^2 + K|\beta_i^n\widehat{\xi}_i^n| \\ &\leq K(|\widehat{\xi}_i^n|^2 + |\widetilde{\xi}_i'^n|^2 + Z_i^n|\widehat{\xi}_i^n|). \end{aligned} \tag{5.26}$$

Moreover in all cases g_n is even, hence g_n' is odd. Therefore (5.19), (5.21), (5.22) and (5.23), together with either (5.24) or (5.25), imply that in all cases (recall $\theta = 2$ in cases (i) and (ii)):

$$\begin{aligned} &\Delta_n \sum_{i=1}^{[t/\Delta_n]} \left|\mathbb{E}_{i-1}^n(g_n(\beta_i'^n) - g_n(\beta_i^n))\right| \\ &\qquad = \mathrm{o}_{Pu}\left(\Delta_n^{1/2}\varepsilon_n^{-(s-1)^+}\right) + \mathrm{O}_{Pu}\left(\Delta_n^{\theta/2}\varepsilon_n^{-\theta(s-1)^+} + \varepsilon_n^{\theta(2-s)/2}\gamma_s(\varepsilon_n)^{\theta/2}\right). \end{aligned}$$

d) The previous estimate plus Lemma 5.2 and (5.18) yield

$$U_t^n = \begin{cases} \mathrm{O}_{Pu}(\Delta_n \varepsilon_n^{-2(s-1)^+} + \Delta_n \varepsilon_n^{-s} + \varepsilon_n^{2-s}\gamma_s(\varepsilon_n)) & \\ \qquad + \mathrm{o}_{Pu}(\sqrt{\Delta_n}\ \varepsilon_n^{-(s-1)^+}) & \text{in case (i)} \\ \mathrm{O}_{Pu}(\Delta_n \varepsilon_n^{-2(s-1)^+} + \Delta_n \alpha_n^2 \varepsilon_n^{-s} + \varepsilon_n^{2-s}\gamma_s(\varepsilon_n)) & \\ \qquad + \mathrm{o}_{Pu}(\sqrt{\Delta_n}\ \varepsilon_n^{-(s-1)^+}) & \text{in case (ii)} \\ \mathrm{O}_{Pu}(\Delta_n^{\theta/2} \varepsilon_n^{-\theta(s-1)^+} + \Delta_n^{1-r/2} \varepsilon_n^{-(s-r)^+} + \varepsilon_n^{\theta(2-s)/2}\gamma_s(\varepsilon_n)^{\theta/2}) & \\ \qquad + \mathrm{o}_{Pu}(\sqrt{\Delta_n}\ \varepsilon_n^{-(s-1)^+}) & \text{in case (iii).} \end{cases}$$

So it remains to choose appropriately $\theta \in [1, 1+r)$ in case (iii) and the sequence ε_n in all three cases.

In case (i) with $s < 1$ take $\varepsilon_n = \sqrt{\Delta_n}$, to obtain $Y_t^n = \mathrm{o}_{Pu}(\sqrt{\Delta_n})$. In case (i) with $s \geq 1$ take $\varepsilon_n = A\sqrt{\Delta_n}$ for some $A > 1$, to obtain for some K not depending on A:

$$\frac{|U_t^n|}{\Delta_n^{1-s/2}} \leq \begin{cases} K\left(\sqrt{\Delta_n} + A^{-1} + A^{2-s}\gamma_s(A\sqrt{\Delta_n})\right) + \mathrm{o}_{Pu}(1) & \text{if } s = 1 \\ K\left(A^{1-s} + B^{2-s}\gamma_s(A\sqrt{\Delta_n})\right) & \text{if } s > 1. \end{cases}$$

Since $\gamma_s(A\sqrt{\Delta_n}) \to 0$ for any A and since K is arbitrarily large, we readily deduce that $U_t^n = \mathrm{o}_{Pu}(\Delta_n^{1-s/2})$, and we have the result.

Next, in case (i) we take $\varepsilon_n = A\Delta_n^{\varpi}$ for some $A > 1$, to obtain for some K not depending on A:

$$|U_t^n| \leq K\Delta_n^{\varpi(2-s)}\left(A^{-s} + A^{2-s}\gamma_s(A\Delta_n^{\varpi})\right) + \mathrm{o}_{Pu}\left(A^{-(s-1)^+}\Delta_n^{1/2-\varpi(s-1)^+}\right),$$

and we conclude as in case (i).

Finally consider case (iii). When $s \leq r$ we choose θ arbitrarily in $(1, 1+r)$ and $\varepsilon_n = \Delta_n^{1/\theta(2-s)}$, and the result in (c) is obvious. When $r < s$, for any given $\theta \in (1, 1+r)$ we choose $\varepsilon_n = \Delta_n^{(2-r)/((2-s)\theta+2(s-r))}$. After a simple (although a bit tedious) computation we deduce

$$U_t^n = \mathrm{O}_{Pu}\left(\Delta_n^{a(\theta,r,s)}\right) + \mathrm{o}_{Pu}\left(\Delta_n^{a'(\theta,r,s)}\right),$$

with $a(\theta,r,s) = \frac{(2-r)(2-s)\theta}{2((2-s)\theta+2(s-r))}$ and $a'(\theta,r,s) = \frac{1}{2}$ if $s \leq 1$ and $a'(\theta,r,s) = \frac{(2-s)(\theta+2(1-r))}{2((2-s)\theta+2(s-r))}$ if $s > 1$. Observe that $a'(\theta,r,s) \geq a(\theta,r,s)$ is $s > 1$ and $\theta < 2$, and that a is increasing in θ. So we should θ as big as possible (with $\theta < 1+r$, though), and another simple computation shows that when $s \leq 1$, then $a(1+r,r,s) > 1/2$ if and only if $\frac{1-\sqrt{5-8s+3s^2}}{2-s} < r \leq 1$. Then again we get the results. $\square$

Lemma 5.6 *Assume (SL-s) and let g_n be as in case (ii) of Lemma 5.5. Then for all $t > 0$ we have*

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}_{i-1}^n\left(\left(g_n(\Delta_i^n X/\sqrt{\Delta_n}) - (\beta_i^n)^2\right)^2\right) \;=\; \mathrm{o}_{Pu}\left(\Delta_n^{4\varpi-2-s\varpi}\right). \qquad (5.27)$$

Proof. First, we have

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}_{i-1}^n\Big(\big((\beta_i^n)^2-g_n(\beta_i^n)\big)^2\Big) \;\leq\; \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}_{i-1}^n\Big((\beta_i^n)^4 1_{\{|\beta_i^n|>\alpha\Delta_n^{\varpi-1/2}\}}\Big)$$
$$\leq\; K\Delta_n^{q((1/2-\varpi)}\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}_{i-1}^n(|\beta_i^n|^{4+q}) \;\leq\; Kt\Delta_n,$$

where the second inequality is valid for all $q>0$ and the third one is obtained by choosing $q=\frac{4}{1-2\varpi}$, and using the boundedness of σ. Therefore is it enough to prove

$$a_n(t) \;:=\; \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}_{i-1}^n\Big(\Big(g_n(\Delta_i^n X/\sqrt{\Delta_n})-g_n(\beta_i^n)\Big)^2\Big) \;=\; \mathrm{o}_{Pu}\big(\Delta_n^{4\varpi-1-s\varpi}\big). \qquad (5.28)$$

With the notation of the proof of Lemma 5.5 we have $\Delta_i^n X=\beta_i^n+\beta_i'^n+\zeta_i^n(4)$, hence by (5.16) we obtain

$$\Big(g_n(\Delta_i^n X/\sqrt{\Delta_n})-g_n(\beta_i^n)\Big)^2 \leq K\alpha_n^2|\beta_i'^n|^2+K\alpha_n^3(|\zeta_i^n(4)\bigwedge\alpha_n).$$

Recall also that $\beta_i'^n=\widehat{\xi}_i^n+\widetilde{\xi}_i'^n$, and we have (5.21), and also (5.22 with $\theta=2$ and $l=0$ and $Z_i^n=1$. Therefore we easily deduce from these two estimates, plus (5.9), that (recall $\alpha_n=\alpha\Delta_n^{\varpi-1/2}$)

$$a_n(t)=\mathrm{O}_{Pu}\Big(\Delta_n^{4\varpi-2}\varepsilon_n^{-s}+\Delta_n^{2\varpi-1}\varepsilon_n^{-(s-1)^+}+\Delta_n^{2\varpi-2}\varepsilon_n^{2-s}\gamma_s(\varepsilon_n)\Big).$$

It remains to choose the sequence ε_n, and we take $\varepsilon_n=A\Delta_n^{\varpi}$ for some $A>1$, which gives

$$a_n(t)\leq K\Delta_n^{4\varpi-2-s\varpi}\Big(A^{-s}+A^{2-s}\gamma_s(A\Delta_n^{\varpi})+\Delta_n^{1-2\varpi+\varpi(s\wedge 1)}A^{((s-1)^+}\Big),$$

and we conclude as in the end of the previous proof. □

5.2 An auxiliary CLT.

We first give a sketchy proof for a result which is essentially known already, and which is a CLT for processes of the form

$$\overline{U}^n(g)_t=\sqrt{\Delta_n}\sum_{i=1}^{[t/\Delta_n]}\Big(g(\beta_i^n)-\rho_i^n(g)\Big). \qquad (5.29)$$

The assumption (SH) below is of course much too strong for the result. For the needs of Theorem 2.16 later one, we give a multidimensional version:

Lemma 5.7 *Assume (SH) and let $g_1,\cdots,g_d$ be continuous even functions in $\mathcal{E}$. The d-dimensional processes $\overline{U}^n$ with components $\overline{U}^n(g_j)$ converge stably in law to a limit $\overline{U}$ with components $\overline{U}_t^j=\sum_{k=1}^d\int_0^t\theta_u^{jk}\,d\overline{W}_u^k$, where the process θ is $(\mathcal{F}_t)$-optional $d\times d$-dimensional processes which satisfies*

$$(\theta_t\theta_t^\star)^{jk} \;=\; \rho_{\sigma_t}(g_jg_k)-\rho_{\sigma_t}(g_j)\rho_{\sigma_t}(g_k). \qquad (5.30)$$

Proof. Observe that $\overline{U}^n_t = \sum_{i=1}^{[t/\Delta_n]} \zeta^n_i$, where ζ^n_i is the $\mathcal{F}_{i\Delta_n}$-measurable variable with components $\zeta^{n,j}_i = \sqrt{\Delta_n}\,\Big(g_j(\beta^n_i) - \rho^n_i(g_j)\Big)$. Moreover $\mathbb{E}^n_{i-1}(\zeta^n_i) = 0$ and $\mathbb{E}^n_{i-1}(\|\zeta^n_i\|^4) \leq K\Delta_n^2$ (because σ_t is bounded). Then a criterion for the stable convergence in law, which can be found in Theorems IX.7.19 and IX.7.28 of [6], gives us the result, provided we have the following two properties:

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}^n_{i-1}(\zeta^{n,j}_i \zeta^{n,k}_i) \;\stackrel{\text{u.c.p.}}{\longrightarrow}\; \int_0^t (\theta_u\theta^\star_u)^{jk}\,du, \tag{5.31}$$

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}^n_{i-1}(\zeta^{n,j}_i\;\Delta^n_i N) \;\stackrel{\text{u.c.p.}}{\longrightarrow}\; 0, \qquad \text{if } N = W \text{ or if } \; N \in \mathcal{N} \tag{5.32}$$

(recall that $\mathcal{N}$ is the set of all bounded $(\mathcal{F}_t)$-martingales which are orthogonal to W). (5.32) follows from (4.6) because g_j is even. Finally $\mathbb{E}^n_{i-1}(\zeta^{n,j}_i \zeta^{n,k}_i))$ equals the right side of (5.30) evaluated at time $u = (i-1)\Delta_n$, and multiplied by Δ_n. Since the right side of (5.30) is a càdlàg function of u, (5.31) follows from Riemann approximation of the integral on the right, and we are done. $\square$

For the purpose of proving Theorem 2.12-(ii) we need more than this lemma. Suppose that we have (SK). Then (2.11) holds with $|\delta(\omega,t,x)| \leq \gamma(x) \leq K$ and $|\widetilde{\delta}(\omega,t,x)| \leq \gamma(x) \leq K$ and $\int \gamma(x)^2 dx < \infty$.

We fix $\varepsilon > 0$ and consider the process $N = 1_E \star \underline{\mu}$, where $E = \{x : \gamma(x) > \varepsilon\}$. Hence N is a Poisson process with parameter the Lebesgue measure of E, say λ. We introduce some notation similar to (3.2), and which depends on ε :

$$\left.\begin{array}{l}
\bullet\; S_1, S_2, \cdots \;\text{ are the successive jump times of } N,\\
\bullet\; I(n,p) = i, \quad S_-(n,p) = (i-1)\Delta_n, \quad S_+(n,p) = i\Delta_n\\
\qquad\qquad \text{on the set } \;\{(i-1)\Delta_n < S_p \leq i\Delta_n\},\\
\bullet\; \alpha_-(n,p) = \frac{1}{\sqrt{\Delta_n}}\Big(W_{S_p} - W_{S_-(n,p)}\Big), \quad \alpha_+(n,p) = \frac{1}{\sqrt{\Delta_n}}\Big(W_{S_+(n,p)} - W_{S_p}\Big)\\
\bullet\; R_p = \Delta X_{S_p},\\
\bullet\; X(\varepsilon)_t = X_t - \sum_{p:\, S_p \leq t} R_p,\\
\bullet\; R'^n_p = \Delta^n_i X(\varepsilon) \;\text{ on the set } \;\{(i-1)\Delta_n < S_p \leq i\Delta_n\},\\
\bullet\; R'_p = \sqrt{\kappa_p}\, U_p\, \sigma_{S_p-} + \sqrt{1-\kappa_p}\, U'_p\, \sigma_{S_p},\\
\bullet\; \Omega_n(T,\varepsilon) \;\text{ is the set of all } \omega \text{ such that each interval } [0,T] \cap ((i-1)\Delta_n, i\Delta_n]\\
\qquad \text{contains at most one } S_p(\omega), \text{ and that } |\Delta^n_i X(\varepsilon)(\omega)| \leq 2\varepsilon \text{ for all } i \leq T/\Delta_n.
\end{array}\right\} \tag{5.33}$$

We also suppose that we have the functions g_j and the processes $\overline{U}^n$ and $\overline{U}$ of Lemma 5.7. Then we have the following result, which is very close to Lemma 6.2 of [7], but with a more involved proof because we want no restriction on the σ-fields $\mathcal{F}_t$.

Lemma 5.8 *Under (SK), the sequences* $\big(\overline{U}^n, (\alpha_-(n,p), \alpha_+(n,p))_{p\geq 1}\big)$ *converge stably in law to* $\big(\overline{U}, (\sqrt{\kappa_p}\; U_p \;, \sqrt{1-\kappa_p}\; U'_p)_{p\geq 1}\big)$ *as* $n \to \infty$.

Proof. *Step 1.* We need to prove the following : for all bounded $\mathcal{F}$-measurable variables Ψ and all bounded Lipschitz functions Φ on the Skorokhod space of d-dimensional functions on $\mathbb{R}_+$ endowed with a distance for the Skorokhod topology, and all $q \geq 1$ and all continuous bounded functions f_p on $\mathbb{R}^2$, and with $A_n = \prod_{p=1}^q f_p(\alpha_-(n,p), \alpha_+(n,p))$, then

$$\mathbb{E}\left(\Psi\ \Phi(\overline{U}^n)\ A_n\right)\ \rightarrow\ \widetilde{\mathbb{E}}(\Psi\ \Phi(\overline{U}))\ \prod_{p=1}^q \widetilde{\mathbb{E}}\Big(f_p(\sqrt{\kappa_p}\ U_p\ ,\sqrt{1-\kappa_p}\ U'_p)\Big). \tag{5.34}$$

Up to substituting Ψ with $\mathbb{E}(\Psi|\mathcal{G})$ in both sides, it is enough to prove this when Ψ is measurable w.r.t. the separable σ-field $\mathcal{G}$ generated by the measure μ and the processes b, σ, W and X.

Step 2. We denote by $\underline{\mu}'$ and $\underline{\mu}''$ (resp. $\underline{\nu}'$ and $\underline{\nu}''$) respectively the restrictions of $\underline{\mu}$ (resp. $\underline{\nu}$) to $\mathbb{R}_+ \times E^c$ and to $\mathbb{R}_+ \times E$. We also denote by $(\mathcal{F}'_t)$ the smallest filtration containing $(\mathcal{F}_t)$ and such that the measure $\underline{\mu}''$ is $\mathcal{F}'_0$-measurable. Then W and $\underline{\mu}'$ are a Wiener process and a Poisson measure with compensator $\underline{\nu}'$, relative to $(\mathcal{F}_t)$ of course, but also to $(\mathcal{F}'_t)$.

Next, for any integer $m \geq 1$ we set $S_p^{m-} = (S_p - 1/m)^+$ and $S_p^{m+} = S_p + 1/m$, and $B_m = \cup_{p\geq 1}(S_p^{m-}, S_p^{m+}]$. The indicator function $1_{B_m}(\omega,t)$ is $\mathcal{F}'_0 \otimes \mathcal{R}_+$-measurable, so the stochastic integral $W(m)_t = \int_0^t 1_{B_m}(u)\ dW_u$ is well defined. We call $(\mathcal{F}'^m_t)$ the smallest filtration containing $(\mathcal{F}'_t)$ and such that the process $W(m)$ is $\mathcal{F}'^m_0$-measurable, and $\Gamma_n(m,t)$ the set of all integers $i \geq 1$ such that $i \leq [t/\Delta_n]$ and that $B_m \cap ((i-1)\Delta_n, i\Delta_n] = \emptyset$, and we introduce the d-dimensional processes $\overline{U}'^n(m)$ and $\overline{U}(m)$ (with θ as in (5.30)) with components :

$$\overline{U}'^{n,j}(m)_t\ =\ \sum_{i\in\Gamma_n(m,t)} \Big(g_j(\beta_i^n) - \rho_i^n(g_j)\Big),\quad \overline{U}^j(m)_t = \sum_{k=1}^d \int_0^t \theta_u^{jk} 1_{B_m^c}(u)\ d\overline{W}_u^k.$$

Again, the integrals above are well defined because $\overline{W}$ is a Brownian motion w.r.t. the smallest filtration containing $(\widetilde{\mathcal{F}}_t)$ and also $\mathcal{F}'^m_0$ at time 0. Furthermore B_m decreases to the union of the graphs of the S_p's, hence $\overline{U}(m)\ \stackrel{\text{u.c.p.}}{\longrightarrow}\ \overline{U}$ as $m \to \infty$. We also have for some $p > 0$ because $g_j \in \mathcal{E}$ and σ is bounded:

$$\begin{aligned}
\mathbb{E}\left(\sup_{s\leq t}|\overline{U}^n(g_j)_s - \overline{U}'^{n,j}(m)_s|^2\right)\ &\leq\ \mathbb{E}\left(\sum_{p\geq 1}\ \sum_{i:\ i\Delta_n\leq t, |i\Delta_n - S_p|\leq 2/m} (g_j(\beta_i^n) - \rho_i^n(g_j))\right)\\
&\leq\ K\mathbb{E}\left(\sum_{p\geq 1}\ \sum_{i:\ i\Delta_n\leq t, |i\Delta_n - S_p|\leq 2/m} (1 + |\Delta_i^n W|^p)\right)\\
&\leq\ \frac{K}{m}\ \mathbb{E}(\sum_{p=1}^\infty 1_{\{S_p\leq t+1\}})\ \leq\ \frac{Kt}{m}.
\end{aligned}$$

Therefore, since Φ is Lipschitz and bounded, it is clearly enough to prove

$$\mathbb{E}\left(\Psi\ \Phi(\overline{U}'^n(m))\ A_n\right) \rightarrow \widetilde{\mathbb{E}}(\Psi\ \Phi(\overline{U}(m)))\ \prod_{p=1}^q \widetilde{\mathbb{E}}\Big(f_p(\sqrt{\kappa_p}\ U_p\ ,\sqrt{1-\kappa_p}\ U'_p)\Big) \tag{5.35}$$

for each m, and for Ψ being $\mathcal{G}$-measurable bounded.

Step 3. In the sequel we fix m, and we introduce a regular version $Q = Q_\omega(.)$ of the probability $\mathbb{P}$ on $(\Omega, \mathcal{G})$, conditional on $\mathcal{F}_0'^m$, and accordingly $\widetilde{Q} = Q \otimes \mathbb{P}'$.

Since $\Delta_i^n W$ is independent of $\mathcal{F}_0'^m$ when $i \in \Gamma_n(m,t)$ it is also standard normal under each Q_ω, and in the proof of Lemma 5.7 we can replace $\mathbb{E}_{i-1}^n$ by the conditional expectation $\mathbb{E}_{Q_\omega}(.|\mathcal{F}_{(i-1)\Delta_n}'^m)$. Moreover B_m^c is a locally finite union of intervals, hence we still have the convergence in (5.31) if the sum on the left is taken over $\Gamma_n(m,t)$ and on the right we plug in $1_{B_m^c}$ in the integral. Hence $\overline{U}'^n(m) \stackrel{\mathcal{L}-(s)}{\longrightarrow} \overline{U}(m)$ under the measure Q_ω, that is

$$\mathbb{E}_{Q_\omega}(\Psi\ \Phi(\overline{U}'^n(m)) \ \to\ \mathbb{E}_{\widetilde{Q}_\omega}(\Psi\ \Phi(\overline{U}(m)). \tag{5.36}$$

Step 4) By construction A_n is $\mathcal{F}_0'^m$-measurable, so the left side of (5.35) is

$$\begin{aligned}\mathbb{E}\Big(A_n\ \mathbb{E}_{Q_.}(\Psi\ \Phi(\overline{U}'^n(m))\Big) &= \mathbb{E}\Big(A_n\ \mathbb{E}_{\widetilde{Q}_.}(\Psi\ \Phi(\overline{U}(m))\Big)\\ &+\mathbb{E}\Big(A_n\ \Big(\mathbb{E}_{Q_.}(\Psi\ \Phi(\overline{U}'^n(m)) - \mathbb{E}_{\widetilde{Q}_.}(\Psi\ \Phi(\overline{U}(m))\Big)\Big).\end{aligned}$$

Since everything above is bounded, the second summand on the right goes to 0 by (5.36), whereas $\Psi' = \mathbb{E}_{\widetilde{Q}_.}(\Psi\ \Phi(\overline{U}(m))$ is another bounded $\mathcal{F}_0'^m$-measurable variable. Hence (5.35) amounts to proving

$$\mathbb{E}\,(\Psi\ A_n)\ \to\ \mathbb{E}(\Psi)\ \prod_{p=1}^q \mathbb{E}\Big(f_p(\sqrt{\kappa_p}\ U_p\ ,\sqrt{1-\kappa_p}\ U_p')\Big),$$

which is exactly $(\alpha_-(n,p), \alpha_+(n,p))_{p\geq 1} \stackrel{\mathcal{L}-(s)}{\longrightarrow} (\sqrt{\kappa_p}\ U_p\ ,\sqrt{1-\kappa_p}\ U_p')_{p\geq 1}$ as $n\to\infty$. But now, this is a consequence of Lemma 6.2 of [7] in a slightly simpler situation, namely we replace α_j^n and β_j^n in that lemma by $\alpha_-(n,p)$ and $\alpha_+(n,p)$ here, respectively, and we do not consider the process $H^{n,\varepsilon}$ in it. Hence we are done. $\square$

Lemma 5.9 *Under the assumptions of Lemma 5.8, the sequences* $\big(\overline{U}^n, (R_p'^n/\sqrt{\Delta_n})_{p\geq 1}\big)$ *converge stably in law to* $\big(\overline{U}, (R_p')_{p\geq 1}\big)$ *as* $n\to\infty$.

Proof. Due to Lemma 5.8 and to the definition of R_p' and the fact that σ is càdlàg, it is clearly enough to prove that for any $p\geq 1$ we have

$$w_p^n\ :=\ R_p'^n/\sqrt{\Delta_n} - \sigma_{S_-(n,p)}\alpha_-(n,p) - \sigma_{S_p}\alpha_+(n,p)\ \stackrel{\mathbb{P}}{\longrightarrow}\ 0. \tag{5.37}$$

We use the notations $\underline{\mu}'$ and $(\mathcal{F}_t')$ of the previous proof. We deduce from (5.3) that

$$X(\varepsilon)_t = X_0 + \int_0^t b'(\varepsilon)_s ds + \int_0^t \sigma_s dW_s + \delta \star (\underline{\mu}' - \underline{\nu}')_t, \tag{5.38}$$

where $b'(\varepsilon)t = b'_t - \int_E \delta(t,x)dx$ and the above stochastic integrals may be taken relative to both filtrations $(\mathcal{F}_t)$ and $(\mathcal{F}'_t)$. In particular $X(\varepsilon)$ satisfies (SH) for the filtration $(\mathcal{F}'_t)$. Similar to $X' = X - X_0 - X^c$, we write $X'(\varepsilon) = X(\varepsilon) - X^c - X_0$. Then

$$w^n_p = \frac{1}{\sqrt{\Delta_n}}\left(\Delta^n_{I(n,p)}X'(\varepsilon) + \int_{S_-(n,p)}^{S_p}(\sigma_u - \sigma_{S(n,p)})dW_s + \int_{S_p}^{S_+(n,p)}(\sigma_u - \sigma_{S_p})dW_s\right).$$

We may write (4.4) for the process $X'(\varepsilon)$ and with the conditional expectations w.r.t. $\mathcal{F}'_{(i-1)\Delta_n}$ instead of $\mathcal{F}_{(i-1)\Delta_n}$. If we additionally use the $\mathcal{F}'_0$-measurability of $I(n,p)$, and if we modify the definition of χ'^n_i for $i = I(n,p)$ as to be the sum of the two stochastic integrals in the previous display, we obtain by taking the expectation :

$$\begin{aligned}\mathbb{E}(\phi_2(w^n_p)) \quad &\leq \quad K\sqrt{\Delta_n} + K\mathbb{E}\Big(\frac{1}{\Delta_n}\int_{S_-(n,p)}^{S_+(n,p)}du\int_{E^c\cap\{x:|\delta(u,x)|\leq\Delta_n^{1/4}\}}\delta(u,x)^2dx\Big)\\ &+K\mathbb{E}\Big(\frac{1}{\Delta_n}\int_{S_-(n,p)}^{S_p}(\sigma_u - \sigma_{S(n,p)})^2du. + \frac{1}{\Delta_n}\int_{S_p}^{S_+(n,p)}(\sigma_u - \sigma_{S_p})^2du\Big)\end{aligned}$$

Since $|\delta| \leq \gamma$ and $\int\gamma(x)^2dx < \infty$ and since σ is càdlàg bounded, we deduce from Lebesgue's theorem that the above goes to 0 as $n\to\infty$, hence $w^n_p \xrightarrow{\mathbb{P}} 0$ and the result is proved. $\square$

5.3 Proof of Theorem 2.8.

We take $f \in \mathcal{E}'_1 \cap C^{0,\nu}$ and $\eta \in (0,\infty]$, with $\eta < \infty$ when f is not bounded. For any $\varepsilon \in (0,\eta/2)$ we have the decomposition $V^n(f) - \overline{H}^n(f\psi_\eta) = \sum_{j=1}^3 Z^n(\varepsilon,j)$, where

$$Z^n(\varepsilon,1) = V^n(f(1-\psi_\varepsilon)) - \overline{H}^n(f\psi_\eta(1-\psi_\varepsilon)),$$

$$Z^n(\varepsilon,2) = Z^n(2) = \sqrt{\Delta_n}\sum_{i=1}^{[t/\Delta_n]}\Big(|\beta^n_i| - |m_1\sigma_{(i-1)\Delta_n}|\Big),$$

$$Z^n(\varepsilon,3)_t = \sum_{i=1}^{[t/\Delta_n]}\Big(\zeta^n_i(\varepsilon) - \mathbb{E}^n_{i-1}(\zeta^n_i(\varepsilon)\Big), \quad \text{with} \quad \zeta^n_i(\varepsilon) = (f\psi_\varepsilon)(\Delta^n_iX) - |\sqrt{\Delta_n}\,\beta^n_i|.$$

First, Theorem 2.3 implies that $Z^n(\varepsilon,1) \xrightarrow{\text{Sk.p.}} \Sigma(f(1-\psi_\varepsilon),\psi_\eta)$ because $f(1-\psi_\varepsilon) \in \mathcal{E}''_r \cap C^{0,\nu}$ for all $r\in(1,2)$. Next, it follows from the Lebesgue dominated convergence theorem for stochastic integrals that $\Sigma(f(1-\psi_\varepsilon),\psi_\eta) \xrightarrow{\text{u.c.p.}} \Sigma(f,\psi_\eta)$ as $\varepsilon\to 0$. Lemma 5.7 implies that $Z^n(2) \xrightarrow{\mathcal{L}-(s)} \sqrt{m_2 - m_1^2}\int_0^t \sigma_u dW'_u$. Hence, due to the properties of the stable convergence in law, it is enough to prove that for all $\rho > 0$,

$$\lim_{\varepsilon\to 0}\ \limsup_n\ \mathbb{P}\left(\sup_{s\leq t}|Z^n(\varepsilon,3)_s| > \rho\right) = 0.$$

But $Z^n(\varepsilon,3)$ is a locally square-integrable martingale w.r.t. the filtration $(\mathcal{F}_{\Delta_n[t/\Delta_n]})_{t\geq 0}$, whose predictable quadratic variation C^n satisfies $C^n_t \leq \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}^n_{i-1}(\zeta^n_i(\varepsilon)^2)$, and for the above it is enough that

$$\lim_{\varepsilon\to 0}\ \limsup_n\ \mathbb{P}\left(C^n_t > \rho\right) = 0$$

for all $\rho > 0$. This is obviously implied by (5.1), and we are done. $\square$

5.4 Proof of Theorem 2.9.

1) We first prove the result under the stronger assumption (SL-s). Let g be a C^2_b and even function in general, or a C^1 even function with $g' \in \mathcal{E}$ when X is continuous. With the notation (5.11) and (5.29), we have

$$\left.\begin{array}{r} \frac{1}{\sqrt{\Delta_n}}\left(\Delta_n V'^n(g)_t - \int_0^t \rho_{\sigma_u}(g)\,du\right) = \overline{U}^n(g)_t + \frac{1}{\sqrt{\Delta_n}}\,U^n(g)_t + M^n_t, \\ \text{where}\quad M^n_t = \sum_{i=1}^{[t/\Delta_n]}\left(\zeta^n_i - \mathbb{E}^n_{i-1}(\zeta^n_i)\right), \end{array}\right\} \tag{5.39}$$

and $\zeta^n_i = \sqrt{\Delta_n}\,\left(g(\Delta^n_i X/\sqrt{\Delta_n}) - g(\beta^n_i)\right)$. Since (SH) holds, we can apply (4.7) for $q=2$ to get

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}^n_{i-1}(|\zeta^n_i|^2)\ \stackrel{\text{u.c.p.}}{\longrightarrow}\ 0. \tag{5.40}$$

By Lenglart's inequality (as in Lemma 4.5) we deduce that $M^n \stackrel{\text{u.c.p.}}{\longrightarrow} 0$. Next, Lemma 5.7 implies that $\overline{U}^n(g)_t \stackrel{\mathcal{L}-(s)}{\longrightarrow} \int_0^t \theta_u\,d\overline{W}_u$, where $\theta_t = \sqrt{\rho_{\sigma_t}(g^2) - \rho_{\sigma_t}(g)^2}$. Hence in view of (5.39) it remains to prove that $\frac{1}{\sqrt{\Delta_n}}\,U^n(g) \stackrel{\text{u.c.p.}}{\longrightarrow} 0$ when $s \leq 1$, and that $\Delta_n^{s/2-1} U^n(g) \stackrel{\text{u.c.p.}}{\longrightarrow} 0$ when $s > 1$: this is given by Lemma 5.5, case (i), and the result is proved.

2) Now we only assume (L-s). A localization procedure, more sophisticated but similar to the one in Lemma 4.6, is described in details in Section 3 of [4]. Namely, we find a sequence (T_p) of stopping times increasing to $+\infty$ and a sequence of processes $(X(p),\sigma(p))$, such that:

- If X satisfies (L-s), then each $X(p)$ satisfies (SL-s).
- We have $(X(p)_t,\sigma(p)_t) = (X_t,\sigma_t)$ for all $t < T_p$, where $\sigma(p)$ denotes the process associated with $X(p)$ in (2.11)-(2.12).

Of course in [4] this localization is done under the additional assumption that $\delta = 0$ (X is continuous), but the presence of a non-vanishing δ does not impair the argument.

On the one hand, Theorem 2.9 holds for each $X(p)$, that is:

$$\begin{array}{l} s \leq 1 \ \Rightarrow\ \frac{1}{\sqrt{\Delta_n}}\left(\Delta_n V'^n(X(p);g)_t - \int_0^t \rho_{\sigma(p)_u}(g)\,du\right) \stackrel{\mathcal{L}-(s)}{\longrightarrow} \int_0^t \theta(p)_u\,d\overline{W}_u \\ s > 1 \ \Rightarrow\ \Delta_n^{1-s/2}\left(\Delta_n V'^n(X(p);g)_t - \int_0^t \rho_{\sigma(p)_u}(g)\,du\right) \stackrel{\text{u.c.p.}}{\longrightarrow} 0, \end{array}$$

where $\theta(p)_t = \sqrt{\rho_{\sigma(p)_t}(g^2) - \rho_{\sigma(p)_t}(g)^2}$. On the other hand bothe the right and the left sides above, written for $(X(p),\sigma(p)$ and also for (X,σ) at time t, agree on the set $\{t < T_p\}$. Since $T_p \to \infty$, we readily deduce Theorem 2.9 for the initial process X.

5.5 Proof of Theorem 2.10.

First, coming back to the localization procedure explained just above, we have that if (L-s) and (H') hold for X, one can find the processes $X(p)$ and $\sigma(p)$ such that (SL-s) *and* (SH') hold. Then the same argument as in the previous proof shows that it is enough to prove Theorem 2.10 under (SL-s) and (SH').

We take $f \in \mathcal{E}_r$ for some $r \in (0,1]$. Instead of (5.39) we have

$$\frac{1}{\sqrt{\Delta_n}}\left(\Delta_n^{1-r/2} V^n(f)_t - m_r \int_0^t c_u^{r/2}\,du\right) = \overline{U}^n(h_r)_t + \frac{1}{\sqrt{\Delta_n}}\,U^n(h_r)_t + M_t^n + N_t^n, \quad (5.41)$$

where M_t^n is like in (5.39) with $\zeta_i^n = \sqrt{\Delta_n}\,\left(h_r(\Delta_i^n X/\sqrt{\Delta_n}) - h_r(\beta_i^n)\right)$, and where $N_t^n = \Delta^{1/2-r/2} V^n(f-h_r)$.

We have $|f-h_r| \leq k$ for some $K \in \mathcal{E}_2''' \cap C^0$, hence $|N^n| \leq \Delta_n^{1-r/2} V^n(k)$ and thus

$$N^n \;=\; \begin{cases} \mathrm{o}_{Pu}(1) & \text{if } \; r<1 \\ \mathrm{O}_{Pu}(1) & \text{if } \; r=1 \end{cases} \quad (5.42)$$

The other terms in (5.41) are treated as in the proof of Theorem 2.9. We have $\overline{U}^n(h_r)_t \stackrel{\mathcal{L}-(s)}{\longrightarrow} \sqrt{m_{2r}-m_r^2}\int_0^t c_u^{r/2}\,d\overline{W}_u$ and $M^n \stackrel{\text{u.c.p.}}{\longrightarrow} 0$ (we can still apply (4.7) with $q=2$ here to get (5.40)). Then in view of (5.41) and (5.42) we readily deduce Theorem 2.10 from Lemma 5.5, case (iii).

Finally suppose that X is continuous: we need to prove the result without (SH'), when $r>1$. Since $\Delta_n^{1-r/2} V^n(h_r) = \Delta_n V'^n(h_r)$, it is a consequence of Theorem 2.9 when $f=h_r$, and for $f \in \mathcal{E}_r$ it remains to prove that $\Delta_n^{1/2-r/2} V^n(f-h_r) \stackrel{\text{u.c.p.}}{\longrightarrow} 0$. Note that $|f-h_r| \leq K h_q$ for some $K>0$ and some $q>r-1$. Since $\mathbb{E}(|\Delta_i^n X|^q) \leq K\Delta_n^{q/2}$ when X is continuous, we get

$$\mathbb{E}\left(\sup_{s\leq t}\left|\Delta_n^{1/2-r/2} V^n(f-h_r)_s\right|\right) \leq K\Delta_n^{1/2-r/2}\sum_{i=1}^{[t/\Delta_n]}\mathbb{E}(|\Delta_i^n X|^q) \leq Kt\Delta_n^{(1-r+q)/2},$$

hence the result.

5.6 Proof of Theorem 2.11.

Here again, as in the two previous proofs, it is enough to prove the result under (SL-s). We have $\varpi \in (0,\frac{1}{2})$ and $\alpha>0$, and we set $g_n = h_2\psi_{\alpha\Delta_n^\varpi}$ and $\overline{g}_n = h_2\psi_{\alpha\Delta_n^\varpi/2}$. Since

$$\Delta_n V'^n(\overline{g}_n) \;\leq\; V''^n(\varpi,\alpha) \;\leq\; \Delta_n V'^n(g_n),$$

hence also $|V''^n(\varpi,\alpha) - C| \leq |\Delta_n V'^n(g_n) - C| + |\Delta_n V'^n(\overline{g}_n) - C|$, it is clearly enough to prove the result for $\Delta_n V'^n(g_n)$ and $\Delta_n V'^n(\overline{g}_n)$ instead of $V''^n(\varpi,\alpha)$, and also that

$$\sqrt{\Delta_n}\;(V'^n(g_n) - V'^n(\overline{g}_n)) \;\stackrel{\text{u.c.p.}}{\longrightarrow}\; 0 \quad (5.43)$$

when $s \leq \frac{4\varpi-1}{2\varpi}$.

Exactly as for (5.39) we observe that,

$$\frac{1}{\sqrt{\Delta_n}}\left(\Delta_n V'^n(g_n) - C\right) = \overline{U}^n(h_2) + \frac{1}{\sqrt{\Delta_n}}\, U^n(g_n) + M^n, \tag{5.44}$$

where M^n_t is like in (5.39) with $\zeta^n_i = \sqrt{\Delta_n}\,\left(g_n(\Delta^n_i X/\sqrt{\Delta_n}) - h_2(\beta^n_i)\right)$. Apply (5.27) to obtain $\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}^n_{i-1}((\zeta^n_i)^2) \;=\; o_{Pu}(\Delta_n^{4\varpi-1-s\varpi})$, hence by Lenglart inequality,

$$M^n \;=\; o_{Pu}(\Delta_n^{2\varpi-1/2-s\varpi/2}). \tag{5.45}$$

Next, Lemma 5.7 implies $\overline{U}^n(h_2)_t \;\stackrel{\mathcal{L}-(s)}{\longrightarrow}\; \sqrt{2}\int_0^t c_u\, d\overline{W}_u$. Hence if we plug Lemma 5.5 for case (ii) and (5.45) into (5.44) we obtain the desired results for $\Delta_n V'^n(g_n)$ (observe that $2\varpi-1/2-s\varpi/2 \geq 0$ if $s \leq (4\varpi-1)/2\varpi$, and otherwise $2\varpi-1/2-s\varpi/2 \geq 2\varpi-1/2-s\varpi$), and of course the same holds for $\Delta_n V'^n(\overline{g}_n)$.

It remains to prove (5.43) when $s \leq \frac{4\varpi-1}{2\varpi}$. We have the same decomposition (5.44) for $\overline{g}_n$, with some $\overline{M}^n$ which also satisfies (5.45), whereas the left side of (5.43) is $M^n - \overline{M}^n$, and this goes u.c.p. to 0 by (5.45), and the proof is finished.

5.7 Proof of Theorem 2.12-(i).

Our first task is to show that (2.15) makes sense, and for further purposes we slightly extend the setting : Take any sequence (T_n) of stopping times whose graphs are pairwise disjoint, and such that $\Delta X_t(\omega) \neq 0$ implies the existence of $n = n(\omega, t)$ such that $t = T_n(\omega)$.

Lemma 5.10 *Under (H), and if $g \in \mathcal{E}''_1$, the increasing process $C(g)$ defined by (2.16) is finite-valued, and the formula (2.15) defines a semimartingale on the extended space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathbb{P}})$, which is a locally square-integrable martingale as soon as the process $C(g)$ above is locally integrable, in which case*

$$\mathbb{E}(Z(g)^2_T) = \mathbb{E}(C(g)_T) \tag{5.46}$$

for any $(\mathcal{F}_t)$-stopping time T. Moreover we have:

a) Conditionally on $\mathcal{F}$, the process $Z(g)$ is a square-integrable martingale with independent increments and predictable bracket $C(g)$, relative to the filtration $(\mathcal{F}\bigvee\widetilde{\mathcal{F}}_t)$, and whose law is completely characterized by the processes X and c and do not depend on the particular sequence (T_n) of stopping times.

b) If further X and σ have no common jumps, then conditionally on $\mathcal{F}$ the process $Z(g)$ is a Gaussian martingale.

In Theorem 2.12 we have a large degree of freedom for defining $Z(f')$, its conditional law w.r.t. the σ-field $\mathcal{F}$ being the only relevant property. This lemma shows that if we change the sequence of stopping times (T_n), subject of course to the property of encompassing all jump times of X, then one changes $Z(f')$ but *not* its conditional law. Note also that $g(0) = 0$ above, so in (2.15) the "part" of T_n for which $\Delta X_{T_n} = 0$ does not come in into the sum, which is consistent with what precedes.

Proof. Among several natural proofs, here is an "elementary" one. Let $g \in \mathcal{E}_1''$, and set $\alpha_n = g(X_{T_n})(c_{T_n-} + \frac{1}{2}\,\Delta c_{T_n})$. We have $g^2 \star \mu_t < \infty$ and c is ω-wise locally bounded, hence $C(g)_t = \sum_n \alpha_n 1_{\{T_n \le t\}} < \infty$ (for $\mathbb{P}$-almost all ω of course).

Fix $\omega \in \Omega$ such that $C(g)(\omega)_t < \infty$ for all $t < \infty$. Under $\mathbb{P}'$, for all n with $T_n(\omega) \le t$ the variables $A_n(\omega) := \sqrt{\kappa_n}\; U_n\; \sigma_{T_n-}(\omega) + \sqrt{1-\kappa_n}\; U_n'\; \sigma_{T_n}(\omega)$ are independent centered with variances $\alpha_n(\omega)$. Then by a standard criterion for convergence of series of independent variables, the formula

$$Z(g)_t(\omega,.) = \sum_{n=1}^{\infty} g(X_{T_n}(\omega))\Big(\sqrt{\kappa_n}\; U_n\; \sigma_{T_n-}(\omega) + \sqrt{1-\kappa_n}\; U_n'\; \sigma_{T_n}(\omega)\Big) 1_{\{T_n(\omega) \le t\}}$$

defines a process $(\omega', t) \mapsto Z(g)(\omega,\omega')_t$ which obviously is a martingale with independent increments. Moreover its predictable bracket is deterministic (that is, it does not depend on ω') and is $C(g)(\omega)$, and it is purely discontinuous and jumps at times $T_n(\omega)$, and the law of the jump at $T_n(\omega) < \infty$ is the law of $g(X_{T_n}(\omega))\Big(\sqrt{\kappa_n}\; U_n\; \sigma_{T_n-}(\omega) + \sqrt{1-\kappa_n}\; U_n'\; \sigma_{T_n}(\omega)\Big)$, which only depends on the processes X and c at point ω. If further X and c have no common jumps, then the law of the jump at $T_n(\omega) < \infty$ is the law of $g(X_{T_n}(\omega))\; \sigma_{T_n}(\omega)\Big(\sqrt{\kappa_n}\; U_n + \sqrt{1-\kappa_n}\; U_n'\Big)$, which is $\mathcal{N}(0, g(X_{T_n}(\omega))^2\; c_{T_n})$. This proves (a) and (b).

Next we consider the properties of $Z(g)$, considered now as a process defined on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\ge 0}, \widetilde{\mathbb{P}})$. Suppose first that $\mathbb{E}(C(g)_{S_p}) < \infty$ for some sequence (S_p) of stopping times increasing to ∞. Then for any $(\mathcal{F}_t)$-stopping time T we have

$$\mathbb{E}(Z(g)_T^2) = \int \mathbb{P}(d\omega) \int \mathbb{P}'(d\omega') Z(g)_{T(\omega)}(\omega,\omega')^2 = \int \mathbb{P}(d\omega) C(g)_{T(\omega)}(\omega),$$

so (5.46) holds and $Z(g)^2_{S_p \bigwedge t}$ is $\widetilde{\mathbb{P}}$-integrable, and for $A \in \widetilde{\mathcal{F}}_t$ and $s \ge 0$ we have

$$\begin{aligned} &\mathbb{E}\left(1_A(Z(g)_{S_p \bigwedge (t+s)} - Z(g)_{S_p \bigwedge t})\right) \\ &= \int \mathbb{P}(d\omega) \int \mathbb{P}(d\omega') 1_A(\omega,\omega')(Z(g)_{S_p(\omega) \bigwedge (t+s)}(\omega,\omega') - Z(g)_{S_p(\omega) \bigwedge t}(\omega,\omega')) = 0, \end{aligned}$$

and thus $Z(g)$ is an $(\widetilde{\mathcal{F}}_t)$-locally square-integrable martingale. In the general case we set $A_n = \{\alpha_n \le 1\}$ and we let $T_n' = T_n$ and $T_n'' = \infty$ on A_n, and $T_n' = \infty$ and $T_n'' = T_n$ on A_n^c. These are stopping times, and we define $Z'(g)_t$ and $Z''(g)_t$ by (2.15), with the sequences (T_n') and (T_n'') respectively. The same analysis as above shows that $Z'(g)$ is an $(\widetilde{\mathcal{F}}_t)$-locally square-integrable martingale, whereas $Z''(g)_t$ is a finite sum and thus as a process it has finite variation. We deduce the semimartingale property of $Z(g) = Z'(g) + Z''(g)$. $\square$

Lemma 5.11 *The claim (i) of Theorem 2.12 holds under (SK) and when f is C^1 and vanishes on a neighborhood of* 0.

Proof. We suppose that $f(x) = 0$ if $|x| \le 2\varepsilon$ for some $\varepsilon > 0$. We use the notation (5.33) associated with this particular ε, so that $|\Delta X_s| \le \varepsilon$ identically if s is not equal to one of the S_p's. Since the derivative f' also vanishes on $[-2\varepsilon, 2\varepsilon]$, we deduce that the process $Z(f')$ has the same law, conditional on $\mathcal{F}$, than the following process:

$$Z_t = \sum_{p:\ S_p \le t} f'(R_p) R'_p.$$

Hence the claim amounts to the stable convergence in law towards Z', for the sequence of processes $Z^n(f)/\sqrt{\Delta_n}$, where $Z^n(f)$ is given by (3.1).

Recall that $V(f) = f \star \mu$. In view of the properties of f we readily check that on the set $\Omega_n(T, \varepsilon)$ we have, for $t \le T$:

$$\begin{aligned} \frac{1}{\sqrt{\Delta_n}} Z^n(f)_t &= \frac{1}{\sqrt{\Delta_n}} \sum_{p:\ S_p \le \Delta_n [t/\Delta_n]} \Big(f(R_p + R'^n_p) - f(R_p) \Big) \\ &= \sum_{p:\ S_p \le \Delta_n [t/\Delta_n]} f'(R_p + \widetilde{R}'^n_p) \frac{R'^n_p}{\sqrt{\Delta_n}}, \end{aligned} \tag{5.47}$$

where $\widetilde{R}^n_p$ is between R_p and $R_p + R'^n_p$. Since $R^n_p \to 0$, hence $\widetilde{R}^n_p \to 0$ as well, and since f' is continuous and $\Omega_n(T, \varepsilon) \to \Omega$, the result is a trivial consequence of Lemma 5.8. $\square$

Now we can prove Theorem 2.12-(i) under (SK). For each $\varepsilon > 0$, we set $f_\varepsilon = f\psi_\varepsilon$, and Lemma 5.11 implies $Z^n(f - f_\varepsilon)/\sqrt{\Delta_n} \stackrel{\mathcal{L}-(s)}{\longrightarrow} Z(f' - f'_\varepsilon)$. On the other hand, $f' \in \mathcal{E}''_1$ and thus $Z(f')$ exists and $C(f'_\varepsilon)_t \to 0$ pointwise (Lebesgue's theorem, notation (2.16)) as $\varepsilon \to 0$, and $C(f'_\varepsilon)_t \le K_t$, so (5.46) and Doob's inequality yield $Z(f'_\varepsilon) \stackrel{\text{u.c.p.}}{\longrightarrow} 0$ and thus $Z(f' - f'_\varepsilon) \stackrel{\text{u.c.p.}}{\longrightarrow} Z(f')$ as $\varepsilon \to 0$. Therefore it remains to prove the following :

$$\lim_{\varepsilon \to 0} \ \limsup_n \ \mathbb{P}\Big(\sup_{s \le t} \Big| Z^n(f_\varepsilon)_t / \sqrt{\Delta_n} \ \Big| > \eta \Big) = 0, \quad \forall \eta > 0,\ \forall t > 0. \tag{5.48}$$

Set

$$k_\varepsilon(x, y) = f_\varepsilon(x + y) - f_\eta(x) - f_\varepsilon(y), \qquad g_\varepsilon(x, y) = k_\varepsilon(x, y) - f'_\varepsilon(x) y. \tag{5.49}$$

For ε small enough the function f_ε is C^2, and $V(f)_\varepsilon) = f_\varepsilon \star \mu$ and (5.3) holds, so Itô's formula yields that $Z^n(f_\varepsilon)/\sqrt{\Delta_n} = A(n, \varepsilon)^{(n)} + M(n, \varepsilon)^{(n)}$, where $M(n, \varepsilon)$ is a locally square-integrable martingale, and with

$$A(n, \varepsilon)_t = \int_0^t a(n, \varepsilon)_u \, du, \quad A'(n, \varepsilon)_t := \langle M(n, \varepsilon, M(n, \varepsilon) \rangle = \int_0^t a'(n, \varepsilon)_u \, du, \tag{5.50}$$

where

$$\left\{ \begin{array}{l} a(n, \varepsilon)_t = \frac{1}{\sqrt{\Delta_n}} \Big(f'_\varepsilon(X_t - X^{(n)}_t) b'_t + \frac{1}{2} \ f''_\eta(X_t - X^{(n)}_t) c_t + \int g_\varepsilon(X_t - X^{(n)}_t, \delta(t, z)) \ dz \Big) \\ a'(n, \varepsilon)_t = \frac{1}{\Delta_n} \ \Big(f'_\varepsilon(X_t - X^{(n)}_t)^2 c_t + \int k_\varepsilon(X_t - X^{(n)}_t, \delta(t, z))^2 \ dz \Big). \end{array} \right.$$

In order to get (5.48), it is enough to prove the following, for all $\eta > 0$, $t > 0$:

$$\lim_{\varepsilon\to 0}\ \limsup_n\ \mathbb{P}\Big(\sup_{s\le t}(|A(n,\varepsilon)_s| + A'(n,\varepsilon)_t > \eta\Big)\ =\ 0. \tag{5.51}$$

Recall that $f(0) = f'(0) = 0$ and $f''(x) = \mathrm{o}(|x|)$ as $x \to 0$, so we have

$$j = 0,1,2 \quad \Rightarrow \quad |f_\varepsilon^{j)}(x)|\ \le\ \alpha_\varepsilon\ (|x| \bigwedge \varepsilon)^{3-j} \tag{5.52}$$

for some α_ε going to 0 as $\varepsilon \to 0$, which implies

$$|k_\eta(x,y)| \le K\alpha_\eta |x|\ |y|, \qquad |g_\eta(x,y)| \le K\alpha_\eta |x|\ y^2. \tag{5.53}$$

Then, in view of (SK), we deduce that $|a(n,\varepsilon)_t| \le K\alpha_\varepsilon |X_t - X_t^{(n)}|/\sqrt{\Delta_n}$ and $a'(n,\varepsilon)_t| \le K\alpha_\varepsilon |X_t - X_t^{(n)}|^2/\Delta_n$. Now, exactly as for (4.2), one readily checks that $\mathbb{E}(|X_{t+s} - X_t|^q) \le K_q s^{q/2}$ for all $q \in (0,2]$ and $s,t \ge 0$, under (SH). Applying this with $q = 1$ and $q = 2$, respectively, gives

$$\mathbb{E}\left(v(A(n,\varepsilon)_T\right) \le KT\alpha_\varepsilon, \qquad \mathbb{E}\left(A'(n,\varepsilon)_T\right) \le KT\alpha_\varepsilon^2,$$

and (5.51) immediately follows because $\alpha_\varepsilon \to 0$.

Finally it remains to prove the result under (K). This is done using the same localization procedure than in Lemma 4.6 or in the proof of Theorems 2.9, and we leave the (easy) details to the reader.

5.8 Proof of Theorem 2.12-(ii).

Before proceeding to the proof itself, we give two preliminary lemmas : the first one is related to Lemma 4.1, the second one is a simple application of Itô's formula.

Lemma 5.12 *Under (SK) there exist increasing functions l_n on $(0,\infty)$ such that*

$$\lim_{\eta\to 0}\ \limsup_n\ l_n(\eta)\ =\ 0, \tag{5.54}$$

and that for all $i,n \in \mathbb{N}$, $\varepsilon,\eta > 0$, we have with $X(\varepsilon)' = X(\varepsilon) - X_0 - X^c$)

$$t \le \Delta_n \quad \Rightarrow \quad \mathbb{E}_{i-1}^n\left(|X(\varepsilon)'_{(i-1)\Delta_n+t} - X(\varepsilon)'_{(i-1)\Delta_n}|^2 \bigwedge \eta^2\right)\ \le\ \Delta_n l_n(\eta). \tag{5.55}$$

Proof. For any $\theta > 0$ we use the decomposition $X(\varepsilon)' = N(\theta) + M(\theta) + B(\theta)$ given in the proof of Lemma 4.1, and also the function $\gamma_2(y)$ given in lemma 5.3. Recall that

$$\begin{cases} \mathbb{P}_{i-1}^n(N(\theta)_{(i-1)\Delta_n+t} - N(\theta)_{(i-1)\Delta_n} \neq 0) \le K\theta^{-2}t, \\ \mathbb{E}_{i-1}^n((M(\theta)_{(i-1)\Delta_n+t} - M(\theta)_{(i-1)\Delta_n}) \le \gamma_2(\theta)t, \\ |B(\theta)_{(i-1)\Delta_n+t} - B(\theta)_{(i-1)\Delta_n}| \le K\theta^{-1}t \end{cases}$$

and K above does not depend on ε. The same argument than in Lemma 4.1 shows that

$$\mathbb{E}^n_{i-1}\left(|X(\varepsilon)'_{(i-1)\Delta_n+t}-X(\varepsilon)'_{(i-1)\Delta_n}|^2\bigwedge\eta^2\right)\ \le\ K\left(\frac{\eta^2\Delta_n}{\theta^2}+\Delta_n\gamma_2(\theta)+\frac{\Delta_n^2}{\theta^2}\right),$$

as soon as $t\le\Delta_n$. So we have (5.55) if we take $l_n(\eta)=K\inf_{\theta\in(0,1]}\Big(\eta^2\theta^{-2}+\gamma_2(\theta)+\Delta_n\theta^{-2}\Big)$, which is obviously increasing in η. Moreover we have (5.54), otherwise there would be an infinite sequence n_k and a number $a>0$ such that $\eta^2\theta^{-2}+\gamma_2(\theta)+\Delta_{n_k}\theta^{-2}\ge a$ for all $\theta\in(0,1]$ and all $\eta>0$, and this contradicts the fact that $\gamma_2(\theta)\to0$ as $\theta\to0$. $\square$

Lemma 5.13 *Under (SK) there is a constant K_0 such that, for each C^2 function g satisfying $g(0)=0$ and $|g'|\le A$ and $|g''|\le A$, we have for all i, n and all $\varepsilon>0$:*

$$t\le\Delta_n\quad\Rightarrow\quad\left\{\begin{array}{l}\left|\mathbb{E}^n_{i-1}\left(g(X(\varepsilon)_{(i-1)\Delta_n+t}-X(\varepsilon)_{(i-1)\Delta_n})\right)\right|\le K_0A\Delta_n,\\ \mathbb{E}^n_{i-1}\left(g(X(\varepsilon)_{(i-1)\Delta_n+t}-X(\varepsilon)_{(i-1)\Delta_n})^2\right)\le K_0(A+A^2)\Delta_n.\end{array}\right.\tag{5.56}$$

If moreover (SL-2) holds we also have

$$t\le\Delta_n\quad\Rightarrow\quad\left\{\begin{array}{l}\left|\mathbb{E}^n_{i-1}\left(c_{(i-1)\Delta_n+t}-c_{(i-1)\Delta_n}\right)\right|\le K\Delta_n,\\ \mathbb{E}^n_{i-1}\left(|c_{(i-1)\Delta_n+t}-c_{(i-1)\Delta_n}|^2\right)\le K\Delta_n.\end{array}\right.\tag{5.57}$$

Proof. By (5.3) and Itô's formula, we have

$$\begin{aligned}g(X(\varepsilon)_{(i-1)\Delta_n+t}-X(\varepsilon)_{(i-1)\Delta_n})\ &=\ \int_{(i-1)\Delta_n}^{(i-1)\Delta_n+t}b(n,i,\varepsilon)_u\ du+\int_{(i-1)\Delta_n}^{(i-1)\Delta_n+t}\sigma(n,i,\varepsilon)_u\ dW_u\\ &\quad+\int_{(i-1)\Delta_n}^{(i-1)\Delta_n+t}\int_{\mathbb{R}}\delta(n,i,\varepsilon)(u,x)(\underline{\mu}-\underline{\nu})(du,dx),\end{aligned}$$

for suitable coefficients easy to compute and which under (SK) satisfy

$$|b(n,i,\varepsilon)_t|\le KA,\quad|\sigma(n,i,\varepsilon)_t|\le KA,\quad|\delta(n,i,\varepsilon)(t,x)|\le KA\gamma(x),$$

uniformly in all arguments (including ω...). Then (5.56) follows in a classical way.

Under (SL-2) the process σ_t satisfies (SK) (except that there are two Brownian motions, but this makes no difference here), so (5.56) applied with $g(x)=x^2$ yields (5.57). $\square$

Now we proceed to the proof of Theorem 2.12-(ii). Upon using the same localization procedure than in the proof of (i), we see that it is enough to prove the result under (SL-2), which we assume thereon. We suppose that $f\in\mathcal{E}_2$, so for $\varepsilon>0$ small enough the function $f_\varepsilon=f\psi_\varepsilon$ is C^∞ and coincides with $h_2\psi_\varepsilon$. We divide the proof into several steps.

Step 1. Fix $\varepsilon>0$. We apply Lemma 5.9 with $d=1$ and $\overline{U}^n=\overline{U}^n(h_2)$ to obtain

$$\left(\overline{U}^n_t,(R'^n_p/\sqrt{\Delta_n})_{p\ge1}\right)\ \stackrel{\mathcal{L}-(s)}{\longrightarrow}\ \left(\sqrt{2}\int_0^t c_u\ d\overline{W}_u,(R'_p)_{p\ge1}\right).$$

On the one hand, the function $f - f_\varepsilon$ satisfies (5.47), so the same argument than in Lemma 5.11 allows to deduce that

$$\Big(\overline{U}^n_t, \frac{1}{\sqrt{\Delta_n}}\, Z^n(f-f_\varepsilon)\Big) \ \stackrel{\mathcal{L}-(s)}{\longrightarrow}\ \Big(\sqrt{2}\int_0^t c_u\, d\overline{W}_u, Z(f'-f'_\varepsilon)\Big). \tag{5.58}$$

On the other hand, suppose for a while that X is continuous. Then both (4.7) and (5.13) hold for $g = h_2$, and so the proof of Theorem 2.9 holds in this case as well, that is

$$\frac{1}{\sqrt{\Delta_n}}\,(\Delta_n V'^n(h_2) - C) - \overline{U}^n = \frac{1}{\sqrt{\Delta_n}}\,(V^n(h_2) - C) - \overline{U}^n \ \stackrel{\text{u.c.p.}}{\longrightarrow} 0.$$

We also have

$$\begin{aligned}\frac{1}{\sqrt{\Delta_n}}\,\mathbb{E}(V^n((1-\psi_\varepsilon)h_2)_t) \ &\leq\ \frac{1}{\sqrt{\Delta_n}}\sum_{i=1}^{[t/\Delta_n]}\mathbb{E}\left(|\Delta_i^n X|^2 1_{\{|\Delta_i^n X|>\varepsilon\}}\right)\\ &\leq\ \frac{1}{\varepsilon^2\sqrt{\Delta_n}}\sum_{i=1}^{[t/\Delta_n]}\mathbb{E}\left(|\Delta_i^n X|^4\right)\ \leq\ Kt\sqrt{\Delta_n}\end{aligned}$$

because $\mathbb{E}(|\Delta_i^n X|^4) \leq K\Delta_n^2$ by (4.2) when X is continuous. Combining these two results yields $\frac{1}{\sqrt{\Delta_n}}\,(V^n(f_\varepsilon) - C) - \overline{U}^n(h_2) \ \stackrel{\text{u.c.p.}}{\longrightarrow} 0$. Now, X is discontinuous, but applying what precedes to X^c yields

$$\frac{1}{\sqrt{\Delta_n}}\,(V^n(X^c; f_\varepsilon) - C) - \sqrt{\Delta_n}\,\overline{U}^n \ \stackrel{\text{u.c.p.}}{\longrightarrow}\ 0.$$

Hence (5.58) holds with $\overline{U}^n$ substituted with $\frac{1}{\sqrt{\Delta_n}}\,(V^n(f_\varepsilon; X^c) - C)$. Since the stochastic integral process in the right side of (5.58) is continuous, we deduce that

$$\frac{1}{\sqrt{\Delta_n}}\,(V^n(X^c; f_\varepsilon)_t - C_t + Z^n(f-f_\varepsilon)_t) \ \stackrel{\mathcal{L}-(s)}{\longrightarrow}\ \sqrt{2}\int_0^t c_u\, d\overline{W}_u + Z(f'-f'_\varepsilon).$$

Furthermore we have $Z(f'-f'_\varepsilon) \ \stackrel{\text{u.c.p.}}{\longrightarrow} Z(f')$ as $\varepsilon \to 0$ (this is like in the previous proof), whereas $V^n(f) - V(f)^{(n)} = Z^n(f - f_\varepsilon) + V^n(f_\varepsilon) - C^{(n)} - f_\varepsilon \star \mu$, and also $V^n(f_\varepsilon)_s = V^n(X(\varepsilon); f_\varepsilon)_s$ for all $s \leq t$ on the set $\Omega(t,\varepsilon)$, which converges to Ω as $\varepsilon \to 0$. Therefore, for obtaining the result it remains to prove that

$$\lim_{\varepsilon\to 0}\ \limsup_n\ \mathbb{P}\Big(\sup_{t\leq T}\Big|Y^n(\varepsilon)_t/\sqrt{\Delta_n}\Big| > \eta\Big) = 0, \quad \forall\eta>0,\ \forall T>0. \tag{5.59}$$

where $Y^n(\varepsilon) = V^n(X(\varepsilon); f_\varepsilon) - V^n(X^c; f_\varepsilon) + C - C^{(n)} - f_\varepsilon \star \mu$.

Step 2. Recall that for ε small enough the function f_ε is C^∞. Then Itô's formula applied with (5.38) yields $Y^n(\varepsilon)/\sqrt{\Delta_n} = A(n,\varepsilon)^{(n)} + M(n,\varepsilon)^{(n)}$, where $M(n,\varepsilon)$ is a locally square-integrable martingale, and we see that (5.50) holds with

$$\begin{aligned}a(n,\varepsilon)_t = \tfrac{1}{\sqrt{\Delta_n}}\Big(&\tfrac{1}{2}\,(f''_\varepsilon(X(\varepsilon)_t - X(\varepsilon)^{(n)}_t) - f''_\varepsilon(X^c_t - (X^c)^{(n)}_t))c_t\\ &+f'_\varepsilon(X(\varepsilon)_t - X(\varepsilon)^{(n)}_t)b'(\varepsilon)_t + \overline{g}_{\varepsilon,t}(X(\varepsilon)_t - X(\varepsilon)^{(n)}_t) + (c_t - c^{(n)}_t)\Big)\\ a'(n,\varepsilon)_t = \tfrac{1}{\Delta_n}\Big(&(f'_\varepsilon(X(\varepsilon)_t - X(\varepsilon)^{(n)}_t) - f'_\varepsilon(X^c_t - (X^c)^{(n)}_t))^2 c_t + \overline{k}_{\varepsilon,t}(X(\varepsilon)_t - X(\varepsilon)^{(n)}_t)\Big),\end{aligned}$$

where we use the notation (5.49) and $E_\varepsilon^c = \{x : \gamma(x) \leq \varepsilon\}$ and

$$\overline{k}_{\varepsilon,t}(x) = \int_{E_\varepsilon^c} k_\varepsilon(x, \delta(t,y))^2 \, dy, \qquad \overline{g}_{\varepsilon,t}(x) = \int_{E_\varepsilon^c} g_\varepsilon(x, \delta(t,y)) \, dy.$$

Here again we are left to proving (5.51). This is more difficult than for (i) of Theorem 2.12, because (5.52) and (5.53) no longer hold. However $A'(n,\varepsilon)$ is increasing, whereas $|a(n,\varepsilon)_t| \leq K$ because of (SL-2) and because the functions f'_ε, f''_ε and $\overline{g}_{\varepsilon,t}$ are obviously bounded by a constant not depending on (ε, t). Hence (5.51) will follow if we prove that for all $\eta > 0$, $t > 0$ we have

$$\lim_{\varepsilon\to 0} \ \limsup_n \ \mathbb{P}\Big(\sup_{s\leq t}(|A(n,\varepsilon)_s^{(n)}| + A'(n,\varepsilon)_t^{(n)} > \eta\Big) \ = \ 0. \tag{5.60}$$

Step 3. We will introduce below some decompositions for $A(n,\varepsilon)^{(n)}$ and $A'(n,\varepsilon)^{(n)}$, namely

$$A(n,\varepsilon)^{(n)} = \sum_{j=1}^{6} D^n(\varepsilon, j), \quad A'(n,\varepsilon)^{(n)} = \sum_{j=7}^{8} D^n(\varepsilon, j), \tag{5.61}$$

where $D^n(\varepsilon,j)_t = \sum_{i=1}^{[t/\Delta_n]} \zeta_i^n(\varepsilon, j)$. Then in order to get (5.60) is it obviously enough to prove that $\lim_{\varepsilon\to 0} \ \limsup_n \ \mathbb{P}\Big(\sup_{s\leq t} |D^n(\varepsilon,j)_s| > \eta\Big) = 0$ for each j. This property obviously holds if

$$\lim_{\varepsilon\to 0} \ \limsup_n \ \mathbb{E}\Big(\sum_{i=1}^{[t/\Delta_n]} |\zeta_i^n(\varepsilon,j)|\Big) \ = \ 0, \tag{5.62}$$

and it also holds if for all $\eta > 0$ we have the following two properties, as $n \to \infty$:

$$\mathbb{E}\Big(\sum_{i=1}^{[t/\Delta_n]} |\mathbb{E}_{i-1}^n(\zeta_i^n(\varepsilon,j))|\Big) \ \to \ 0, \qquad \mathbb{E}\Big(\sum_{i=1}^{[t/\Delta_n]} |\zeta_i^n(\varepsilon,j)|^2\Big) \ \to \ 0. \tag{5.63}$$

Step 4. Before deriving (5.60) we state a number of properties of the functions f_ε, $\overline{g}_{\varepsilon,t}$ and $\overline{k}_{\varepsilon,t}$ and their derivatives. These properties are elementary, although sometimes tedious to derive, and they are based on the fact that f_ε is C^∞ for ε small enough, and $f_\varepsilon(x) = x^2$ when $|x| \leq \varepsilon$ and $f_\varepsilon(x) = 0$ when $|x| \geq 2\varepsilon$; we also use (SK) for (5.67) below, where the notation $\gamma_2(y)$ of Lemma 5.3 is used. Here is the list of those properties :

$$|f_\varepsilon^{(l)}(x)| \leq K_l \varepsilon^{2-l} 1_{\{|x|\leq 2\varepsilon\}}, \tag{5.64}$$

$$|f'_\varepsilon(x+y) - f'_\varepsilon(x)|^2 \leq K(x^4/\varepsilon^2 + y^2 \bigwedge \varepsilon^2), \tag{5.65}$$

$$|f''_\varepsilon(x+y) - f''_\varepsilon(x)| \leq K(x^2 + y^2)/\varepsilon^2, \tag{5.66}$$

$$|\overline{g}_{\varepsilon,t}(x)| \leq K(x^2/\varepsilon^2 + |x|/\varepsilon), \qquad \overline{k}_{\eta,t}(x) \leq K x^2 \gamma_2(\varepsilon), \tag{5.67}$$

$$l = 1,2 \quad \Rightarrow \quad |\overline{g}_{\eta,t}^{(l)}(x)| \leq K\eta^{-l}, \tag{5.68}$$

$$|\overline{g}_{\eta,t}(x) - \overline{g}_{\eta,s}(x)| \le K|x| \int_{E_\varepsilon^c} |\delta(t,z) - \delta(s,z)| \; \gamma(z) \; dz \le K|x|\gamma_2(\varepsilon). \tag{5.69}$$

Step 5. Now, recalling that the right limit $b'(\varepsilon)_{t+}$ of $b'(\varepsilon)$ exists, and with $\overline{g}_{\varepsilon,t+}(x) = \int g_\varepsilon(x, \delta_+(t,y)) \; dy$ (see the notation before (5.5), we set

$$\begin{array}{rcl}
\zeta_i^n(\varepsilon,1) & = & \frac{1}{\sqrt{\Delta_n}} \int_{(i-1)\Delta_n}^{i\Delta_n} (c_t - c_{(i-1)\Delta_n}) \; dt \\
\zeta_i^n(\varepsilon,2) & = & \frac{1}{2\sqrt{\Delta_n}} \int_{(i-1)\Delta_n}^{i\Delta_n} \Big(f''_\varepsilon(X(\varepsilon)_t - X(\varepsilon)_{(i-1)\Delta_n}) - f''_\varepsilon(X_t^c - X^c_{(i-1)\Delta_n}) \Big) \; c_t \; dt \\
\zeta_i^n(\varepsilon,3) & = & \frac{1}{\sqrt{\Delta_n}} \; b'(\varepsilon)_{(i-1)\Delta_n +} \int_{(i-1)\Delta_n}^{i\Delta_n} f'_\varepsilon(X(\varepsilon)_t - X(\varepsilon)_{(i-1)\Delta_n}) \; dt \\
\zeta_i^n(\varepsilon,4) & = & \frac{1}{\sqrt{\Delta_n}} \; \int_{(i-1)\Delta_n}^{i\Delta_n} f'_\varepsilon(X(\varepsilon)_t - X(\varepsilon)_{(i-1)\Delta_n})(b'(\varepsilon)_t - b'(\varepsilon)_{(i-1)\Delta_n +}) \; dt \\
\zeta_i^n(\varepsilon,5) & = & \frac{1}{\sqrt{\Delta_n}} \; \int_{(i-1)\Delta_n}^{i\Delta_n} \overline{g}_{\varepsilon,(i-1)\Delta_n +}(X(\varepsilon)_t - X(\varepsilon)_{(i-1)\Delta_n}) \; dt \\
\zeta_i^n(\varepsilon,6) & = & \frac{1}{\sqrt{\Delta_n}} \; \int_{(i-1)\Delta_n}^{i\Delta_n} \Big(\overline{g}_{\varepsilon,t} - \overline{g}_{\varepsilon,(i-1)\Delta_n +} \Big) (X(\varepsilon)_t - X(\varepsilon)_{(i-1)\Delta_n}) \; dt \\
\zeta_i^n(\varepsilon,7) & = & \frac{1}{\Delta_n} \; \int_{(i-1)\Delta_n}^{i\Delta_n} \Big(f'_\eta(X(\varepsilon)_t - X(\varepsilon)_{(i-1)\Delta_n}) - f'_\varepsilon(X_t^c - X^c_{(i-1)\Delta_n}) \Big)^2 \; c_t \; dt \\
\zeta_i^n(\varepsilon,8) & = & \frac{1}{\Delta_n} \; \int_{(i-1)\Delta_n}^{i\Delta_n} \overline{k}_{\varepsilon,t}(X(\varepsilon)_t - X(\varepsilon)_{(i-1)\Delta_n}) \; dt
\end{array}$$

With these variables, it is easy to check that (5.61) holds. Hence it remains to prove that for each $j = 1, \ldots, 8$ we have either (5.62) or (5.63). This is the aim of the following lemma, which will end our proof.

Lemma 5.14 *We have (5.62) for $j = 2, 4, 6, 7, 8$.*

Proof. Recalling $X(\varepsilon) = X_0 + X^c + X(\varepsilon)'$, we deduce from (5.66) that

$$|\zeta_i^n(\varepsilon,2)| \le \frac{K}{\varepsilon^2 \sqrt{\Delta_n}} \int_{(i-1)\Delta_n}^{i\Delta_n} \Big((X_t^c - X^c_{(i-1)\Delta_n})^2 + (X(\varepsilon)'_t - X(\varepsilon)'_{(i-1)\Delta_n})^2 \Big) \; dt.$$

Applying (5.56) with $g(x) = x$ to the two processes X^c and $X(\varepsilon)'$ readily gives $\mathbb{E}_{i-1}^n(|\zeta_i^n(\varepsilon,2)|) \le K\Delta_n^{3/2}/\varepsilon^2$, and (5.62) follows.

In a similar way, (5.65) gives

$$|\zeta_i^n(\varepsilon,7)| \le \frac{K}{\Delta_n} \int_{(i-1)\Delta_n}^{i\Delta_n} \Big(\varepsilon^{-2}(X_t^c - X^c_{(i-1)\Delta_n})^4 + \Big((X(\varepsilon)'_t - X(\varepsilon)'_{(i-1)\Delta_n})^2 \bigwedge \varepsilon^2 \Big) \Big) \; dt.$$

Applying the well known fact that $\mathbb{E}_{i-1}^n\Big((X_t^c - X^c_{(i-1)\Delta_n})^4 \Big) \le Kt^2$, and (5.55), we deduce

$$\mathbb{E}_{i-1}^n(|\zeta_i^n(\varepsilon,7)|) \le K \left(\frac{\Delta_n^2}{\varepsilon^2} + \Delta_n l_n(\varepsilon) \right).$$

Then we readily deduce (5.62) from (5.54).

Use (5.69) and (5.67), together with (5.56) again and Cauchy-Schwarz for $j = 6$, to get

$$\mathbb{E}^n_{i-1}(|\zeta(\varepsilon,6)|) + \mathbb{E}^n_{i-1}(|\zeta(\varepsilon,8)|) \leq K\Delta_n \gamma_2(\varepsilon).$$

Since $\gamma_2(\varepsilon) \to 0$ as $\varepsilon \to 0$, we deduce (5.62) for $j = 6, 8$.

Finally consider the case $j = 6$. We use (5.64) and (5.56) once more, plus Cauchy-Schwarz, to get (with $b'^{(n)}_+$ being the process associated with $(b'_{t+}$ by (2.1):

$$\begin{aligned}
\mathbb{E}\left(\sum_{i=1}^{[t/\Delta_n]} |\zeta^n_i(\varepsilon,6)|\right) &\leq \frac{1}{\sqrt{\Delta_n}}\, \mathbb{E}^n_{i-1}\left(\int_0^t |X(\varepsilon)_s - X(\varepsilon)^{(n)}_s|\, |b'(\varepsilon)_s - b'(\varepsilon)^{(n)}_{s+}|\, ds\right) \\
&\leq \frac{1}{\sqrt{\Delta_n}}\left(\mathbb{E}^n_{i-1}\left(\int_0^t |X(\varepsilon)_s - X(\varepsilon)^{(n)}_s|^2\, ds\right) \mathbb{E}^n_{i-1}\left(\int_0^t |b'(\varepsilon)_s - b'(\varepsilon)^{(n)}_{s+}|^2\, ds\right)\right)^{1/2} \\
&\leq \left(\mathbb{E}^n_{i-1}\left(\int_0^t |b'(\varepsilon)_s - b'(\varepsilon)^{(n)}_{s+}|^2\, ds\right)\right)^{1/2}
\end{aligned}$$

where the last inequality comes from (5.56). The last term above goes to 0 because $b'(\varepsilon)_s - b'(\varepsilon)^{(n)}_{s+}$ goes pointwise to 0 and is bounded: therefore we have (5.62) for $j = 6$. □

Lemma 5.15 *We have (5.63) for* $j = 1, 3, 5$.

Proof. Note that $\zeta^n_i(\varepsilon,1) = \zeta^n_i(1)$ does not depend on ε. Then (5.63) for $j = 1$ readily follows from (5.57).

Next, use (5.56) for the function f'_ε and (5.64) and the boundedness of b' to obtain

$$\left|\mathbb{E}^n_{i-1}(\zeta^n_i(\varepsilon,3))\right| \leq \frac{K\Delta_n^{3/2}}{\varepsilon}, \qquad \mathbb{E}^n_{i-1}(\zeta^n_i(\varepsilon,3)^2) \leq \frac{K\Delta_n^2}{\varepsilon^2},$$

and we readily deduce (5.63) for $j = 3$. The same argument also show (5.63) for $j = 5$: we use (5.68), and (5.56) with the function $\overline{g}_{\varepsilon,(i-1)\Delta_n+}$ (this function is random, but $\mathcal{F}_{(i-1)\Delta_n}$-measurable and with uniform bounds on its derivatives, so (5.56) applies in this case). □

5.9 Proof of Theorem 2.16.

Due to all what precedes, the proof is very easy: on the one hand, Lemma 5.9 is already multidimensional. On the other hand, the way Theorems 2.9, 2.10, 2.11 and 2.12 are deduced from Lemma 5.9 can be carried over separately for each component, in the multidimensional case. Therefore Theorem 2.16 holds.

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